

ON THE RATE OF CONVERGENCE OF STRONG EULER APPROXIMATION FOR SDES DRIVEN BY LEVY PROCESSES

R. MIKULEVIČIUS AND FANHUI XU

ABSTRACT. A SDE driven by an α -stable process, $\alpha \in [1, 2)$, with Lipschitz continuous coefficient and β -Hölder drift is considered. The existence and uniqueness of a strong solution is proved when $\beta > 1 - \alpha/2$ by showing that it is L_p -limit of Euler approximations. The L_p -error (rate of convergence) is obtained for a nondegenerate truncated and non-truncated driving process. The rate in the case of Lipschitz continuous coefficients is derived as well.

CONTENTS

1. Introduction	1
2. Notation and Auxiliary Results	5
2.1. Notation	5
2.2. Auxiliary Results	6
3. Proof of Main Results	16
3.1. Proof of Proposition 3	16
3.2. Proof of Proposition 4	20
3.3. Proof of Proposition 1	21
3.4. Proof of Proposition 2	29
4. Appendix	34
References	37

1. INTRODUCTION

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space, and $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,1]}$ be a filtration of σ -algebras satisfying the usual conditions. Let $N(dt, dy)$ be adapted Poisson point measure on $[0, 1) \times \mathbf{R}_0^d$ ($\mathbf{R}_0^d = \mathbf{R}^d \setminus \{0\}$) such that

$$\mathbf{E}N(dt, dy) = \rho(y) \frac{dydt}{|y|^{d+\alpha}},$$

Date: June 22, 2016.

1991 *Mathematics Subject Classification.* 60H10, 60H35, 41A25.

Key words and phrases. Strong solutions, Levy processes, strong approximation.

where $\rho(y)$ is a bounded measurable function, and $\alpha \in [1, 2)$. We consider the following stochastic differential equation (SDE) in time interval $[0, 1]$

$$(1.1) \quad X_t = x_0 + \int_0^t b(X_s) ds + \int_0^t G(X_{s-}) dL_s.$$

The drift coefficient $b : \mathbf{R}^d \rightarrow \mathbf{R}^d$ is a bounded function of β -Hölder continuity in whole space with $\beta \in (0, 1]$, $G(x)$, $x \in \mathbf{R}^d$, is a Lipschitz continuous bounded $d \times d$ -matrix, and for $t \in [0, 1]$,

$$\begin{aligned} L_t &= \int_0^t \int y q(ds, dy), \text{ if } \alpha \in (1, 2), \\ L_t &= \int_0^t \int_{|y|>1} y N(ds, dy) + \int_0^t \int_{|y|\leq 1} y q(ds, dy), \text{ if } \alpha = 1, \end{aligned}$$

where

$$q(dt, dy) = N(dt, dy) - \rho(y) \frac{dy dt}{|y|^{d+\alpha}}$$

is a martingale measure. We will need the following assumptions for ρ .

- S**(c_0). (i) $\rho(y) \geq c_0$, $y \in \mathbf{R}_0^d$ for some $c_0 > 0$;
(ii) $\rho(\lambda y) = \rho(y)$ for all $\lambda > 0$, $y \in \mathbf{R}_0^d$, i.e., ρ is a 0-homogeneous function;
(iii)

$$(1.2) \quad \rho(-y) = \rho(y), y \in \mathbf{R}_0^d, \text{ if } \alpha = 1.$$

We are going to study the Euler approximation to (1.1) defined as

$$(1.3) \quad X_t^n = x_0 + \int_0^t b(X_{\pi_n(s)}^n) ds + \int_0^t G(X_{\pi_n(s)}^n) dL_s,$$

where $\pi_n(s) = k/n$ if $k/n \leq s < (k+1)/n$, $n = 1, 2, \dots$, $k = 0, \dots, n-1$. Note that the driving process L_t does not have α -moment.

Sometimes in (1.1) L_t is replaced by its truncation

$$L_t^0 = \int_0^t \int_{|y|\leq 1} y q(dr, dy), t \in [0, 1],$$

i.e., the following equation and the accompanying Euler approximation are considered instead,

$$(1.4) \quad Y_t = x_0 + \int_0^t b(Y_s) ds + \int_0^t G(Y_{s-}) dL_s^0, t \in [0, 1],$$

and

$$(1.5) \quad Y_t^n = x_0 + \int_0^t b(Y_{\pi_n(s)}^n) ds + \int_0^t G(Y_{\pi_n(s)}^n) dL_s^0, t \in [0, 1].$$

This case would be the other concern of our note. It is well-known that the truncated driving process L_t^0 has all moments.

In [10], the existence and uniqueness of strong solutions to (1.1) was considered by assuming $G = I_d$, the $d \times d$ -identity matrix, and with L_t being nondegenerate α -stable symmetric, $\alpha \in [1, 2)$, $\beta > 1 - \alpha/2$. The pathwise

uniqueness for (1.1) was proved by applying Gronwall's lemma and using the elliptic version of the Kolmogorov equation and regularity of its solution, to represent the Hölder drift $b(x)$ by an expression which is "Lipshitz". This approach, "Itô-Tanaka trick", was inspired by considerations in [4], see the infinite dimensional generalization in [2] for $G = I$ and $L = W$ being Wiener, or a finite dimensional generalization (using parabolic backward Kolmogorov equations) in [3], again with $G = I_d, L = W$, and b having some integrability properties.

On the other hand, in [9] a truncated equation (1.4) and its Euler approximation (1.5) were considered with $G = I_d, \rho = 1$. Using the same Itô-Tanaka trick and assuming that a strong solution Y_t exists with $\alpha + \beta > 2, \beta \in (0, 1)$, the rate of strong convergence was derived. It was proved in [9] that

$$(1.6) \quad \mathbf{E} \left[\sup_t |Y_t^n - Y_t|^p \right] \leq C_p \begin{cases} n^{-1} & \text{if } p \geq 2/\beta, \\ n^{-p\beta/2} & \text{if } 2 \leq p < 2/\beta. \end{cases}$$

In this note, using Itô-Tanaka trick again, we derive the rate of convergence of Euler approximations for both (1.1) and (1.4). We show that, under the imposed assumptions, X^n, Y^n are Cauchy sequences whose limits solve (1.1) and (1.4) respectively.

For (1.1), the following holds. Note that only the moments $p < \alpha$ exist in this case.

Proposition 1. *Let $\alpha \in [1, 2), \mathbf{S}(c_0)$ hold, $\beta \in (0, 1)$ and $\beta > 1 - \alpha/2$. Assume $b \in C^\beta(\mathbf{R}^d)$, G is bounded Lipshitz and $|\det G(x)| \geq c_0 > 0, x \in \mathbf{R}^d$, i.e. G is uniformly nondegenerate. Let for some $c_1 > 0$,*

$$|\rho(y) - \rho(z)| \leq c_1 |y - z|^\beta \text{ for all } |y| = |z| = 1.$$

Then there is a unique strong solution to (1.1). Moreover for each $p \in (0, \alpha)$, there is C depending on $d, \alpha, \beta, b, G, p, \rho$ such that

$$\mathbf{E} \left[\sup_t |X_t^n - X_t|^p \right] \leq C n^{-p\beta/\alpha}.$$

For (1.4) we derive the following statement which extends and improves the results in ([9]), see (1.6).

Proposition 2. *Let $\alpha \in [1, 2), \mathbf{S}(c_0)$ hold, $\beta \in (0, 1)$ and $\beta > 1 - \alpha/2$. Assume $b \in C^\beta(\mathbf{R}^d)$, G is bounded Lipshitz and $|\det G(x)| \geq c_0 > 0, x \in \mathbf{R}^d$, i.e. G is uniformly nondegenerate. Let for some $c_1 > 0$,*

$$|\rho(y) - \rho(z)| \leq c_1 |y - z|^\beta \text{ for all } |y| = |z| = 1.$$

Then there is a unique strong solution to (1.4). Moreover for each $p \in (0, \infty)$, there is C depending on $d, \alpha, \beta, b, G, p, \rho$ such that

$$\mathbf{E} \left[\sup_t |Y_t^n - Y_t|^p \right] \leq C \begin{cases} n^{-p\beta/\alpha} & \text{if } 0 < p < \alpha/\beta, \\ (n/\ln n)^{-1} & \text{if } p = \alpha/\beta, \\ n^{-1} & \text{if } p > \alpha/\beta. \end{cases}$$

In both statements above, L and G are nondegenerate (Assumption $\mathbf{S}(c_0)$ holds). On the other hand, if b and G are Lipschitz continuous, then there exists a unique solution to (1.1) (see Theorem 6.2.3, [1]) with any bounded nonnegative ρ . In this note, we use direct estimates of stochastic integrals to derive the convergence rate in the Lipschitz, possibly completely degenerate, case.

The following statement holds for all Lipschitz case of (1.1).

Proposition 3. *Let $\alpha \in [1, 2)$, ρ be nonnegative bounded. Assume b and G are bounded Lipschitz functions. Then*

(i) *For each $p \in (0, \alpha)$, there is C depending on d, α, b, G, p, ρ such that*

$$\begin{aligned} \mathbf{E} \left[\sup_t |X_t^n - X_t|^p \right] &\leq C (n/\ln n)^{-p/\alpha} \text{ if } 0 < p < \alpha \in (1, 2), \\ \mathbf{E} \left[\sup_t |X_t^n - X_t|^p \right] &\leq C \left[n/(\ln n)^2 \right]^{-p} \text{ if } 0 < p < \alpha = 1. \end{aligned}$$

(ii) *If $\alpha = 1$, and $\rho(y) = \rho(-y)$, $y \in \mathbf{R}^d$, then there is C depending on d, α, b, G, p, ρ such that*

$$\mathbf{E} \left[\sup_t |X_t^n - X_t|^p \right] \leq C (n/\ln n)^{-p} \text{ if } 0 < p < \alpha = 1.$$

We derive the following rate of convergence in all Lipschitz case for (1.4).

Proposition 4. *Let $\alpha \in [1, 2)$, ρ be nonnegative bounded. Assume b and G are bounded Lipschitz functions. Then*

(i) *For each $p \in (0, \alpha)$, there is C depending on d, α, b, G, p, ρ such that*

$$\mathbf{E} \left[\sup_t |Y_t^n - Y_t|^p \right] \leq C \begin{cases} (n/\ln n)^{-p/\alpha} & \text{if } 0 < p < \alpha \in (1, 2), \\ \left[n/(\ln n)^2 \right]^{-p} & \text{if } 0 < p < \alpha = 1, \\ \left[n/(\ln n)^2 \right]^{-1} & \text{if } p = \alpha, \\ n^{-1} & \text{if } p > \alpha \end{cases}$$

(ii) *If $\alpha = 1$, and $\rho(y) = \rho(-y)$, $y \in \mathbf{R}^d$, then there is C depending on d, α, b, G, p, ρ such that*

$$\mathbf{E} \left[\sup_t |Y_t^n - Y_t|^p \right] \leq C (n/\ln n)^{-p} \text{ if } 0 < p < \alpha = 1.$$

The rates above are in agreement with the subtle results obtained in [5] for (1.1) in the case $d = 1, b = 0, G \in C^3$.

An obvious consequence of Proposition 3 is

Corollary 1. *Let $\alpha \in [1, 2)$, ρ be nonnegative bounded. Assume b and G are bounded Lipschitz functions. Then*

(i) there is C depending on d, α, b, G, p, ρ such that for each $\varphi \in C^\beta(\mathbf{R}^d)$, $t \in [0, 1]$,

$$|\mathbf{E}\varphi(X_t) - \mathbf{E}\varphi(X_t^n)| \leq C |\varphi|_\beta (n/\ln n)^{-\beta/\alpha} \text{ if } \alpha \in (1, 2),$$

$$|\mathbf{E}\varphi(X_t) - \mathbf{E}\varphi(X_t^n)| \leq C |\varphi|_\beta [n/(\ln n)^2]^{-\beta} \text{ if } \alpha = 1.$$

(ii) If $\alpha = 1$, and $\rho(y) = \rho(-y)$, $y \in \mathbf{R}^d$, then there is C depending on d, α, b, G, p, ρ such that for each $\varphi \in C^\beta(\mathbf{R}^d)$, $t \in [0, 1]$,

$$|\mathbf{E}\varphi(X_t) - \mathbf{E}\varphi(X_t^n)| \leq C |\varphi|_\beta (n/\ln n)^{-\beta}.$$

Our note is organized as follows. In section 2, notation is introduced, primary analytic tools are discussed and some auxiliary results are presented. In section 3, we prove Propositions 1-4.

2. NOTATION AND AUXILIARY RESULTS

2.1. Notation. $\mathbf{R}_0^d := \mathbf{R}^d \setminus \{0\}$. Denote $H_T = [0, T] \times \mathbf{R}^d$, $0 \leq T \leq 1$. I_d is the $d \times d$ -identity matrix.

For any $x, y \in \mathbb{R}^d$, we write

$$(x, y) = \sum_{i=1}^d x_i y_i, \quad |x| = (x, x)^{1/2}.$$

For a function $u = u(t, x)$ on H , we denote its partial derivatives by $\partial_t u = \partial u / \partial t$, $\partial_i u = \partial u / \partial x_i$, $\partial_{ij}^2 u = \partial^2 u / \partial x_i \partial x_j$, and denote its gradient with respect to x by $\nabla u = (\partial_1 u, \dots, \partial_d u)$ and $D^{|\gamma|} u = \partial^{|\gamma|} u / \partial x_1^{\gamma_1} \dots \partial x_d^{\gamma_d}$, where $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbf{N}^d$ is a multi-index. Meanwhile, we write

$$\begin{aligned} |u|_0 &= \sup_{t, x} |u(t, x)|, \\ [u]_\beta &= \sup_{t, x, h \neq 0} \frac{|u(t, x+h) - u(t, x)|}{|h|^\beta} \quad \text{if } \beta \in (0, 1), \\ [u]_\beta &= \sup_{t, x, h \neq 0} \frac{|u(t, x+h) - u(t, x)|}{|h|} \quad \text{if } \beta = 1. \end{aligned}$$

For $\beta = [\beta] + \{\beta\} > 0$, where $[\beta] \in \mathbf{N}$ is the greatest integer that is less than or equal to β and $\{\beta\} \in (0, 1)$, $C^\beta(H_T)$ denotes the space of measurable functions u on H_T such that the norm

$$|u|_\beta = \sum_{|\gamma| \leq [\beta]} |D^\gamma u|_0 + \sum_{|\gamma| = [\beta]} [D^\gamma u]_{\{\beta\}} < \infty.$$

Analogous definitions apply to functions on \mathbf{R}^d , and $C^\beta(\mathbf{R}^d)$ denotes the corresponding function space.

For a $d \times d$ matrix $G(x)$ on \mathbf{R}^d , we define its norm to be the operator norm, i.e.,

$$|G(x)| := \sup_{y \in \mathbf{R}^d, |y|=1} |G(x)y|,$$

and

$$\|G\| := \sup_{x \in \mathbf{R}^d} |G(x)|.$$

In our note, $\|G\|$ is assumed to be finite and that implies each entry $|G_{ij}|_0 \leq \|G\|$.

Because Lipschitz continuity implies differentiability almost everywhere, we write $|\nabla G|_\infty$ to denote the Lipschitz constant of G , even if G is not specified to be differentiable.

At last, $C = C(\cdot, \dots, \cdot)$ denotes constants depending only on quantities appearing in parentheses, but it may represent different values in different contexts.

2.2. Auxiliary Results.

2.2.1. Backward Kolmogorov equations in Hölder classes. We will rely on some results about backward Kolmogorov equations. For convenience, we summarize assumptions that will be needed as follows:

A(K, c_0). (i) **S**(c_0) holds and for the same c_0 ,

$$|\det G(x)| \geq c_0, x \in \mathbf{R}^d;$$

(ii) There is a constant K such that

$$\|G\| + |\nabla G|_\infty \leq K, \quad 0 \leq \rho(y) \leq K, y \in \mathbf{R}^d.$$

Define for $v \in C_0^\infty(\mathbf{R}^d)$, $x \in \mathbf{R}^d$,

$$(2.1) \quad Lv(x) = \int_{|y| \leq 1} [v(x + G(x)y) - v(x) - (\nabla v(x) \cdot G(x)y)] \rho(y) \frac{dy}{|y|^{d+\alpha}}.$$

Proposition 5. Let $\alpha \in [1, 2)$, $\mu \in (0, 1)$, $\tilde{b} = (\tilde{b}^k)_{1 \leq k \leq d}$ with $\tilde{b}^k \in C^\mu(\mathbf{R}^d)$, $|\tilde{b}^k|_\mu \leq K \forall k$, and Assumption **A**(K, c_0) hold. Let

$$|\rho(y) - \rho(z)| \leq K |y - z|^\beta \text{ for all } |y| = |z| = 1.$$

Then for any $f \in C^\mu(H_1)$, there exists a unique solution $u \in C^{\alpha+\mu}(H_1)$ to the parabolic equation

$$(2.2) \quad \begin{aligned} \partial_t u(t, x) &= Lu(t, x) + \tilde{b}(x) \cdot \nabla u(t, x) + f(t, x), \quad (t, x) \in H_1, \\ u(0, x) &= 0, \quad x \in \mathbf{R}^d. \end{aligned}$$

Moreover, there is a constant $C = C(\alpha, \mu, d, K, c_0)$ such that

$$|u|_{\alpha+\mu} \leq C |f|_\mu,$$

and for all $s \leq t \leq 1$,

$$|u(t, \cdot) - u(s, \cdot)|_{\frac{\alpha}{2}+\mu} \leq C (t - s)^{1/2} |f|_\mu.$$

Proof. We apply Theorem 4 in [8] with $\mathcal{L} = A + B$, where

$$\begin{aligned} Au(t, x) &= \int [u(t, x + G(x)y) - u(t, x) \\ &\quad - (\nabla u(t, x) \cdot G(x)y) \chi_\alpha(y)] \rho(y) \frac{dy}{|y|^{d+\alpha}}, \\ Bu(t, x) &= \bar{b}(x) \cdot \nabla u(t, x) \\ &\quad - \int_{|y|>1} [u(t, x + G(x)y) - u(t, x)] \rho(y) \frac{dy}{|y|^{d+\alpha}}, \quad (t, x) \in H_1, \end{aligned}$$

$\chi_\alpha(y) = 1$ if $\alpha \in (1, 2)$, $\chi_\alpha(y) = \chi_{\{|y| \leq 1\}}(y)$ if $\alpha = 1$, and

$$\bar{b}(x) = \tilde{b}(x) + 1_{\alpha \in (1, 2)} G(x) \int_{|y|>1} y \rho(y) \frac{dy}{|y|^{d+\alpha}}, \quad x \in \mathbf{R}^d.$$

Using the symmetry assumption on ρ and changing variables of integration, we see that

$$Au(t, x) = \int [u(t, x + y) - u(t, x) - (\nabla u(t, x) \cdot y) \chi_\alpha(y)] m(x, y) \frac{dy}{|y|^{d+\alpha}},$$

where for $x \in \mathbf{R}^d, y \in \mathbf{R}_0^d$,

$$m(x, y) = \frac{\rho(G^{-1}(x)y)}{|\det G(x)| \left| G^{-1}(x) \frac{y}{|y|} \right|^{d+\alpha}} := \tilde{m}(x, y) \rho(G^{-1}(x)y).$$

First we verify assumptions of Theorem 4 in [8] for $m(x, y)$. Obviously,

$$|G(x)y| \leq K|y|, \quad x, y \in \mathbf{R}^d,$$

which implies $|y| \leq K \left| G(x)^{-1}y \right|$ and thus $\left| G^{-1}(x) \frac{y}{|y|} \right| \geq 1/K, x \in \mathbf{R}^d, y \in \mathbf{R}_0^d$. Therefore,

$$|m(x, y)| \leq \frac{K^{d+\alpha+1}}{c_0}, \quad x \in \mathbf{R}^d, y \in \mathbf{R}_0^d.$$

On the other hand, it's obvious that $\det G(x)$ is bounded and Lipshitz with $c_0 \leq |\det G(x)| \leq K^d d!$, which implies both $\frac{1}{|\det G(x)|}$ and $\left| G^{-1}(x) \frac{y}{|y|} \right| = \left| \frac{\text{adj}(G(x))}{\det G(x)} \frac{y}{|y|} \right|$ are Lipshitz in x uniformly over y . With

$$(2.3) \quad K^{-1} \leq \left| G^{-1}(x) \frac{y}{|y|} \right| \leq \frac{K^{d-1} (d-1)! d^{3/2}}{c_0} =: c_1, \quad x \in \mathbf{R}, y \in \mathbf{R}_0^d,$$

we can conclude $\tilde{m}(x, y)$ is Lipschitz uniformly over y . Meanwhile, recall that ρ is μ -Hölder continuous and 0-homogeneous. Hence

$$\begin{aligned} & \frac{|\rho(G^{-1}(x+h)y) - \rho(G^{-1}(x)y)|}{|h|^\mu} \\ &= \frac{|\rho(G^{-1}(x+h)\frac{y}{|y|}) - \rho(G^{-1}(x)\frac{y}{|y|})|}{|G^{-1}(x+h)\frac{y}{|y|} - G^{-1}(x)\frac{y}{|y|}|^\mu} \cdot \frac{|G^{-1}(x+h)\frac{y}{|y|} - G^{-1}(x)\frac{y}{|y|}|^\mu}{|h|^\mu} \\ &\leq K |\nabla(G^{-1})|_\infty^\mu, \end{aligned}$$

and therefore $m(x, y)$ is μ -continuous in x uniformly over y .

When $\alpha = 1$, according to (1.2),

$$\begin{aligned} \int_{r < |y| \leq 1} y m(x, y) \frac{dy}{|y|^{d+\alpha}} &= \int_{r < |y| \leq 1} \frac{y \rho(G^{-1}(x)y)}{|\det G(x)| |G^{-1}(x)\frac{y}{|y|}|^{d+\alpha}} \frac{dy}{|y|^{d+\alpha}} \\ &= \int_{r < |y| \leq 1} \frac{-y \rho(G^{-1}(x)y)}{|\det G(x)| |G^{-1}(x)\frac{y}{|y|}|^{d+\alpha}} \frac{dy}{|y|^{d+\alpha}} = 0. \end{aligned}$$

Note that, there is $c_2 = c_2(c_0, \alpha, K, d)$ such that $m(x, y) \geq c_2$, $\forall x \in \mathbf{R}^d, \forall y \in \mathbf{R}_0^d$. Then, **Assumption A** in Theorem 4 of [8] is satisfied.

Let $U = \{y : |y| > 1\}$, $U_1 = \{y : |y| \leq 1\}$, and $c(x, y) = G(x)y$ if $|y| > 1$, $c(x, y) = 0$ otherwise. Then $Bu(t, x)$ can be written as

$$\begin{aligned} Bu(t, x) &= \bar{b}(x) \cdot \nabla u(t, x) - \int_U [u(t, x + c(x, y)) - u(t, x) \\ &\quad - (\nabla u(t, x) \cdot c(x, y)) 1_{U_1}(y)] \rho(y) \frac{dy}{|y|^{d+\alpha}}. \end{aligned}$$

By (2.3), $|y| \leq c_1 |G(x)y|$ for all $x, y \in \mathbf{R}^d$, thus $|c(x, y)| \geq c_1^{-1}$ for all $x, y \in \mathbf{R}^d$. Then by choosing $\varepsilon < c_1^{-1}$, we have

$$\int_{|c(x, y)| \leq \varepsilon} |c(x, y)|^\alpha \rho(y) \frac{dy}{|y|^{d+\alpha}} = 0, \quad \forall x \in \mathbf{R}^d.$$

Hence, **Assumption B1** of Theorem 4 in [8] holds.

We might as well set $K > 1$. Now, for $|h| \leq 1$,

$$\begin{aligned} & \int_{|y| > 1} [|c(x, y) - c(x+h, y)| \wedge 1] \rho(y) \frac{dy}{|y|^{d+\alpha}} \\ &\leq K^2 \int_{|y| > 1} [|h| |y| \wedge 1] \frac{dy}{|y|^{d+\alpha}} = K^2 \int_{|h||y| > |h|} [|h| |y| \wedge 1] \frac{dy}{|y|^{d+\alpha}} \\ &= K^2 |h|^\alpha \int_{|z| > |h|} [|z| \wedge 1] \frac{dz}{|z|^{d+\alpha}} \leq C |h| (1 + 1_{\alpha=1} |\ln |h||) \end{aligned}$$

for some $C = C(\alpha, K, d)$, Therefore **Assumption B2** of Theorem 4 in [8] is satisfied and our statement holds. \square

Now, consider the backward Kolmogorov equation

$$(2.4) \quad \begin{aligned} \partial_t v(t, x) + \tilde{b}(x) \cdot \nabla v(t, x) + Lv(t, x) &= f(x), \quad (t, x) \in H_T, \\ v(T, x) &= 0, \quad x \in \mathbf{R}^d, \end{aligned}$$

where L is defined as (2.1). If u solves (2.2) in H_1 with $f = f(x)$, $x \in \mathbf{R}^d$, then $v(t, x) = u(T - t, x)$, $T - 1 \leq t \leq T$, $x \in \mathbf{R}^d$, solves (2.4) with $T \in [0, 1]$. The following statement is an obvious consequence of Proposition 5.

Corollary 2. *Let $\alpha \in [1, 2)$, $\mu \in (0, 1)$, $\tilde{b} = (\tilde{b}^k)_{1 \leq k \leq d}$ with $\tilde{b}^k \in C^\mu(\mathbf{R}^d)$, $|\tilde{b}^k|_\mu \leq K \forall k$, and Assumption **A**(K, c_0) hold. Let*

$$|\rho(y) - \rho(z)| \leq K |y - z|^\beta \text{ for all } |y| = |z| = 1.$$

Then for any $f \in C^\mu(\mathbf{R}^d)$ and $T \in [0, 1]$, there exists a unique solution $v \in C^{\alpha+\mu}(H_T)$ to (2.4). Moreover, there is a constant $C = C(\alpha, \mu, d, K, c_0)$, independent of T , such that

$$|v|_{\alpha+\mu} \leq C |f|_\mu,$$

and for all $0 \leq s \leq t \leq T$,

$$|v(t, \cdot) - v(s, \cdot)|_{\frac{\alpha}{2}+\mu} \leq C (t - s)^{1/2} |f|_\mu.$$

2.2.2. Some estimates of stochastic integrals and driving processes. We present here some stochastic integral estimates related to stable type point measures. Let $\mathcal{P} = \mathcal{P}(\mathbb{F})$ be predictable σ -algebra on $[0, 1) \times \Omega$.

Let $F : [0, 1) \times \Omega \times \mathbf{R}_0^d \rightarrow \mathbf{R}^m$ be a $\mathcal{P} \times \mathcal{B}(\mathbf{R}_0^d)$ -measurable vector function,

$$F = F_r(y) = (F_r^i(y))_{1 \leq i \leq m}, r \in [0, 1), y \in \mathbf{R}_0^d,$$

such that for any $T \in [0, 1)$ a.s.,

$$(2.5) \quad \int_0^T \int_{|y| \leq 1} |F_r(y)|^2 \rho(y) \frac{dy dr}{|y|^{d+\alpha}} < \infty.$$

Let $0 \leq S \leq T \leq 1$. Consider the stochastic process

$$U_t = \int_S^t \int_{|y| \leq 1} F_r(y) q(dr, dy), t \in [S, T].$$

Note U_t is well defined because of (2.5).

The following estimates hold.

Lemma 1. *Let $\alpha \in [1, 2)$, $p \in (\alpha, \infty)$, $0 \leq \rho(y) \leq K$, $y \in \mathbf{R}^d$. Assume there is a predictable nonnegative process \bar{F}_r , $r \in [S, T]$, such that*

$$|F_r(y)| \leq \bar{F}_r |y|, r \in [S, T], y \in \mathbf{R}^d.$$

Then there is $C = C(d, p, \alpha, K)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |U_t|^p \right] \leq C \mathbf{E} \int_S^T |\bar{F}_r|^p dr.$$

Proof. If $p \geq 2$, then by Lemma 10(i) (e.g. Lemma 4.1 in [6]),

$$\begin{aligned}
 (2.6) \quad & \mathbf{E} \left[\sup_{S \leq t \leq T} |U_t|^p \right] \\
 & \leq C \mathbf{E} \left[\left(\int_S^T \int_{|y| \leq 1} |\bar{F}_r y|^2 \frac{dy dr}{|y|^{d+\alpha}} \right)^{p/2} + \int_S^T \int_{|y| \leq 1} |\bar{F}_r y|^p \frac{dy dr}{|y|^{d+\alpha}} \right] \\
 & \leq C \mathbf{E} \int_S^T |\bar{F}_r|^p dr.
 \end{aligned}$$

If $p \in (\alpha, 2)$, then by Burkholder-Davis-Gundy (BDG) inequality, see Remark 1,

$$\begin{aligned}
 (2.7) \quad & \mathbf{E} \left[\sup_{S \leq t \leq T} |U_t|^p \right] \leq C \mathbf{E} \left[\left(\int_S^T \int_{|y| \leq 1} |\bar{F}_r y|^2 N(dr, dy) \right)^{p/2} \right] \\
 & \leq C \mathbf{E} \left[\int_S^T \int_{|y| \leq 1} |\bar{F}_r y|^p N(dr, dy) \right] \leq C \mathbf{E} \int_S^T |\bar{F}_r|^p dr.
 \end{aligned}$$

□

Lemma 2. Let $0 \leq \rho(y) \leq K, y \in \mathbf{R}^d$. Assume there is a predictable nonnegative process $\bar{F}_r, r \in [S, T]$, such that

$$|F_r(y)| \leq \bar{F}_r |y|, r \in [S, T], y \in \mathbf{R}^d.$$

(i) Let $\alpha \in (1, 2), p \in (0, \alpha)$. Then there is $C = C(d, p, \alpha, K)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |U_t|^p \right] \leq C \left(\mathbf{E} \left[\int_S^T |\bar{F}_r|^\alpha dr \right] \right)^{p/\alpha}.$$

(ii) Let $\alpha \in [1, 2), \bar{F}_r \leq M$ a.s. for some constant $M > 0$ and $\mathbf{E} \int_S^T \bar{F}_r^\alpha dr < 1$. Then there is $C = C(d, \alpha, M, K)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |U_t|^\alpha \right] \leq C \mathbf{E} \int_S^T \bar{F}_r^\alpha dr \left[1 + \left| \ln \left(\mathbf{E} \int_S^T \bar{F}_r^\alpha dr \right) \right| \right].$$

Proof. For any $\varepsilon > 0$,

$$\begin{aligned}
 U_t &= \int_S^t \int_{\bar{F}_r |y| \leq \varepsilon, |y| \leq 1} \cdots + \int_S^t \int_{\bar{F}_r |y| > \varepsilon, |y| \leq 1} \cdots \\
 &:= U_t^1 + U_t^2, t \in [S, T].
 \end{aligned}$$

Let $0 < p \leq \alpha \in [1, 2)$. By Remark 1 (Corollary II in [7]),

$$\begin{aligned}
 \mathbf{E} \left[\sup_{S \leq t \leq T} |U_t^1|^p \right] &\leq C \mathbf{E} \left[\left(\int_S^T \int_{|\bar{F}_r y| \leq \varepsilon, |y| \leq 1} |F_r(y)|^2 \frac{dy dr}{|y|^{d+\alpha}} \right)^{p/2} \right] \\
 (2.8) \quad &\leq C \mathbf{E} \left[\left(\int_S^T \int_{|\bar{F}_r y| \leq \varepsilon} |\bar{F}_r y|^2 \frac{dy dr}{|y|^{d+\alpha}} \right)^{p/2} \right] \\
 &\leq C \varepsilon^{(1-\alpha/2)p} \left(\mathbf{E} \int_S^T \bar{F}_r^\alpha dr \right)^{p/2}.
 \end{aligned}$$

Let $p \in [1, 2)$. Then by BDG inequality, Remark 1,

$$\begin{aligned}
 \mathbf{E} \left[\sup_{S \leq t \leq T} |U_t^2|^p \right] &\leq C \mathbf{E} \left[\left(\int_S^T \int_{|\bar{F}_r y| > \varepsilon, |y| \leq 1} |F_r(y)|^2 N(dr, dy) \right)^{p/2} \right] \\
 (2.9) \quad &\leq C \mathbf{E} \left[\int_S^T \int_{|\bar{F}_r y| > \varepsilon, |y| \leq 1} |\bar{F}_r y|^p \frac{dy dr}{|y|^{d+\alpha}} \right].
 \end{aligned}$$

If $p \in [1, \alpha)$, $\alpha \in (1, 2)$, then

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |U_t^2|^p \right] \leq C \varepsilon^{-(\alpha-p)} \mathbf{E} \int_S^T \bar{F}_r^\alpha dr.$$

Taking $\varepsilon = \left(\mathbf{E} \int_S^T \bar{F}_r^\alpha dr \right)^{1/\alpha}$ and combining with (2.8),

$$(2.10) \quad \mathbf{E} \left[\sup_{S \leq t \leq T} |U_t|^p \right] \leq C \left(\mathbf{E} \int_S^T \bar{F}_r^\alpha dr \right)^{p/\alpha}.$$

If $p \in (0, 1)$, $\alpha \in (1, 2)$, then by Hölder inequality and (2.10),

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |U_t|^p \right] \leq \left(\mathbf{E} \sup_{S \leq t \leq T} |U_t| \right)^p \leq C \left(\mathbf{E} \int_S^T \bar{F}_r^\alpha dr \right)^{p/\alpha}.$$

If $p = \alpha \in [1, 2)$, then, according to (2.9),

$$\begin{aligned}
 \mathbf{E} \left[\sup_{S \leq t \leq T} |U_t^2|^p \right] &\leq C \mathbf{E} \left[\int_S^T \int_{\varepsilon < |\bar{F}_r y| \leq M} |\bar{F}_r y|^\alpha \frac{dy dr}{|y|^{d+\alpha}} \right] \\
 &\leq C (1 + |\ln \varepsilon|) \mathbf{E} \int_S^T \bar{F}_r^\alpha dr.
 \end{aligned}$$

Taking $\varepsilon = \mathbf{E} \int_S^T \bar{F}_r^\alpha dr$ and combining with (2.8), we see that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |U_t|^\alpha \right] \leq C \mathbf{E} \int_S^T \bar{F}_r^\alpha dr \left[1 + |\ln \left(\mathbf{E} \int_S^T \bar{F}_r^\alpha dr \right)| \right].$$

□

Lemma 3. *Let $0 \leq \rho(y) \leq K, y \in \mathbf{R}^d$. Assume there is a predictable nonnegative process $\bar{F}_r, r \in [S, T]$, such that*

$$|F_r(y)| \leq \bar{F}_r |y|, r \in [S, T], y \in \mathbf{R}^d.$$

(i) *Let $\alpha = 1, p \in (0, 1)$, and $\bar{F}_r \leq M$ a.s. for some constant $M > 0$. Then there is $C = C(d, p, \alpha, M, K)$ such that*

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |U_t|^p \right] \leq C \left(\mathbf{E} \int_S^T \bar{F}_r^\alpha dr \right)^p \left[1 + \left| \ln \left(\mathbf{E} \int_S^T \bar{F}_r^\alpha dr \right) \right| \right]^p.$$

(ii) *Let $\alpha = 1, p \in (0, 1)$. Assume $\rho(y) = \rho(-y), y \in \mathbf{R}^d$. Suppose there exists a predictable $m \times d$ matrix valued function $H_r, r \in [S, T]$, such that a.s.*

$$|F_r(y) - H_r y| \leq M \bar{F}_r |y|^{1+\beta'}, r \in [S, T], |y| \leq 1,$$

for some constants $M > 0, \beta' > 0$. Then there is $C = C(d, p, \alpha, M, K)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |U_t|^p \right] \leq C \left(\mathbf{E} \int_S^T |\bar{F}_r| dr \right)^p.$$

Proof. (i) Let $\alpha = 1, p \in (0, 1)$. By Hölder inequality,

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |U_t|^p \right] \leq \left(\mathbf{E} \sup_{S \leq t \leq T} |U_t| \right)^p,$$

and the estimate follows by Lemma 2(ii).

(ii) For $\varepsilon > 0$, we decompose

$$\begin{aligned} U_t &= \int_S^t \int_{\bar{F}_r |y| \leq \varepsilon, |y| \leq 1} \cdots + \int_S^t \int_{\bar{F}_r |y| > \varepsilon, |y| \leq 1} \cdots \\ &:= U_t^1 + U_t^2, t \in [S, T]. \end{aligned}$$

Let $0 < p < 1$. By Remark 1 (Corollary II in [7]), there is $C = C(K, d, p)$ such that

$$\begin{aligned} (2.11) \quad \mathbf{E} \left[\sup_{S \leq t \leq T} |U_t^1|^p \right] &\leq C \mathbf{E} \left[\left(\int_S^T \int_{|\bar{F}_r y| \leq \varepsilon} |\bar{F}_r y|^2 \frac{dy dr}{|y|^{d+1}} \right)^{p/2} \right] \\ &\leq C \varepsilon^{p/2} \left(\mathbf{E} \int_S^T \bar{F}_r dr \right)^{p/2}. \end{aligned}$$

We decompose further

$$\begin{aligned} U_t^2 &= \int_S^t \int_{\bar{F}_r |y| > \varepsilon, |y| \leq 1} F_r(y) N(dr, dy) \\ &\quad + \int_S^t \int_{\bar{F}_r |y| > \varepsilon, |y| \leq 1} (H_r y - F_r(y)) \rho(y) \frac{dy dr}{|y|^{d+1}} \\ &:= U_t^{21} + U_t^{22}, t \in [S, T]. \end{aligned}$$

Now,

$$\begin{aligned} \mathbf{E} \left[\sup_{S \leq t \leq T} |U_t^{21}|^p \right] &\leq \mathbf{E} \int_S^T \int_{\bar{F}_r |y| > \varepsilon, |y| \leq 1} |\bar{F}_r y|^p N(dr, dy) \\ &\leq C \mathbf{E} \int_S^T \int_{\bar{F}_r |y| > \varepsilon} |\bar{F}_r y|^p \frac{dr}{|y|^{d+1}} \leq C \varepsilon^{-(1-p)} \mathbf{E} \int_S^T \bar{F}_r dr, \end{aligned}$$

and

$$\begin{aligned} \mathbf{E} \left[\sup_{S \leq t \leq T} |U_t^{22}|^p \right] &\leq C \mathbf{E} \left(\int_S^T \int_{\bar{F}_r |y| > \varepsilon, |y| \leq 1} |F_r(y) - H_r y| \frac{dy}{|y|^{d+1}} dr \right)^p \\ &\leq C \mathbf{E} \left(\int_S^T \int_{|y| \leq 1} \bar{F}_r |y|^{1+\beta'} \frac{dy}{|y|^{d+1}} dr \right)^p \\ &\leq C \mathbf{E} \left[\left(\int_S^T \bar{F}_r dr \right)^p \right] \leq C \left(\mathbf{E} \int_S^T \bar{F}_r dr \right)^p. \end{aligned}$$

Combining these estimates with (2.11) and taking $\varepsilon = \mathbf{E} \int_S^T \bar{F}_r dr$, we see that for $\alpha = 1, p \in (0, 1)$, there is $C = C(\alpha, d, p, K, M)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |U_t|^p \right] \leq C \left(\mathbf{E} \int_S^T \bar{F}_r dr \right)^p.$$

□

Again, let $F : [0, 1] \times \Omega \times \mathbf{R}_0^d \rightarrow \mathbf{R}^m$ be a $\mathcal{P} \times \mathcal{B}(\mathbf{R}^d)$ -measurable vector function,

$$F = F_r(y) = (F_r^i(y))_{1 \leq i \leq m}, r \in [0, 1], y \in \mathbf{R}_0^d,$$

such that for any $T \in [0, 1]$ a.s.

$$(2.12) \quad \int_0^T \int_{|y| > 1} |F_r(y)| \rho(y) \frac{dy dr}{|y|^{d+\alpha}} < \infty \text{ if } \alpha \in [1, 2].$$

Let $0 \leq S \leq T \leq 1$. Consider the stochastic process

$$Z_t = \int_S^t \int_{|y| > 1} F_r(y) N(dr, dy), t \in [S, T].$$

Note Z_t is well defined because of (2.12).

Later we will need the following estimates as well.

Lemma 4. *Let $\alpha \in [1, 2], p \in (0, \alpha), 0 \leq \rho(y) \leq K, y \in \mathbf{R}^d, 0 \leq S \leq T \leq 1$. Assume there is a predictable nonnegative process $\bar{F}_r, r \in [S, T]$, such that*

$$|F_r(y)| \leq \bar{F}_r |y|, r \in [S, T], y \in \mathbf{R}^d.$$

Then there is $C = C(d, p, \alpha, K)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |Z_t|^p \right] \leq C \mathbf{E} \int_S^T |\bar{F}_r|^p dr.$$

Proof. Let $p \in (0, 1)$. Then, according to Remark 3,

$$\begin{aligned} \mathbf{E} \left[\sup_{S \leq t \leq T} |Z_t|^p \right] &\leq C \mathbf{E} \left[\int_S^T \int_{|y|>1} |F_r(y)|^p \rho(y) \frac{dy dr}{|y|^{d+\alpha}} \right] \\ &\leq C \mathbf{E} \int_S^T \bar{F}_r^p dr. \end{aligned}$$

Let $p \in [1, \alpha)$, $\alpha \in (1, 2)$. By Lemma 10(ii),

$$\begin{aligned} \mathbf{E} \left[\sup_t |Z_t|^p \right] &\leq C \mathbf{E} \left[\left(\int_S^T \int_{|y|>1} |\bar{F}_r y| \frac{dy dr}{|y|^{d+\alpha}} \right)^p + \int_S^T \int_{|y|>1} |\bar{F}_r y|^p \frac{dy dr}{|y|^{d+\alpha}} \right] \\ &\leq C \mathbf{E} \left[\left(\int_S^T \bar{F}_r dr \right)^p + \int_S^T \bar{F}_r^p dr \right] \leq C \mathbf{E} \int_S^T \bar{F}_r^p dr. \end{aligned}$$

□

We now apply Lemmas 1-3 to estimate

$$L_t^0 = \int_0^t \int_{|y| \leq 1} y q(dr, dy), t \in [0, 1].$$

Lemma 5. *Let $0 \leq \rho(y) \leq K$.*

(i) *There is $C = C(\alpha, d, p, K)$ such that for all $t \in [0, 1]$,*

$$\begin{aligned} \mathbf{E} [|L_t^0|^p] &\leq Ct \text{ if } p > \alpha \in [1, 2), \\ \mathbf{E} [|L_t^0|^p] &\leq Ct(1 + |\ln t|) \text{ if } p = \alpha \in [1, 2), \\ \mathbf{E} [|L_t^0|^p] &\leq Ct^{p/\alpha} \text{ if } p < \alpha \in (1, 2), \end{aligned}$$

and

$$\mathbf{E} [|L_t^0|^p] \leq Ct^p (1 + |\ln t|)^p, p < \alpha = 1.$$

(ii) *Let $\alpha = 1$ and $\rho(y) = \rho(-y)$, $y \in \mathbf{R}^d$. There is $C = C(d, p, K)$ such that for all $t \in [0, 1]$,*

$$\mathbf{E} [|L_t^0|^p] \leq Ct^p \text{ if } p < \alpha = 1.$$

Proof. These estimates are obvious consequences of Lemmas 1 - 3 when they are applied to $F_r(y) = y, y \in \mathbf{R}^d$. □

Now we estimate

$$L_t = L_t^0 + \int_0^t \int_{|y|>1} y N(dr, dy) - 1_{\alpha \in (1, 2)} t \int_{|y|>1} y \rho(y) \frac{dy}{|y|^{d+\alpha}},$$

$t \in [S, T]$.

Lemma 6. *Let $0 \leq \rho(y) \leq K$.*

(i) *For each $p \in (0, \alpha)$ there is $C = C(\alpha, d, p, K)$ such that for all $t \in [0, 1]$,*

$$\mathbf{E} [|L_t|^p] \leq Ct^{p/\alpha} \text{ if } \alpha \in (1, 2),$$

and

$$\mathbf{E} [|L_t|^p] \leq Ct^p (1 + |\ln t|)^p \text{ if } \alpha = 1.$$

(ii) If $\alpha = 1$, $\rho(y) = \rho(-y)$, $y \in \mathbf{R}^d$. Then for each $p \in (0, \alpha)$ there is $C = C(p, d, K)$ such that

$$\mathbf{E}[|L_t|^p] \leq Ct^p, t \in [0, 1].$$

(iii) If $\alpha \in [1, 2)$, then there is $C = C(\alpha, d, K)$ such that

$$\mathbf{E}[|L_t|^\alpha \wedge 1] \leq Ct(1 + |\ln t|), t \in [0, 1].$$

Proof. The estimates in (i)-(ii) are obvious consequences of Lemmas 5 and 4 applied to $F_r(y) = y$, $y \in \mathbf{R}^d$. We prove (iii) only.

Let

$$\begin{aligned} V_t &= \int_0^t \int_{|y|>1} y N(dr, dy), \\ P_t &= 1_{\alpha \in (1, 2)} \int_0^t \int_{|y|>1} y \rho(y) \frac{dy dr}{|y|^{d+\alpha}}, \end{aligned}$$

i.e., $L_t = L_t^0 + V_t - P_t$, $t \in [0, 1]$. According to Lemma 5, there is $C = C(\alpha, d, K)$ so that

$$\mathbf{E}[|L_t^0|^\alpha] \leq Ct(1 + |\ln t|), t \in [0, 1]$$

Now,

$$\begin{aligned} |V_t|^\alpha \wedge 1 &= \int_0^t \int_{|y|>1} [(|V_{r-} + y|^\alpha \wedge 1) - (|V_{r-}|^\alpha \wedge 1)] N(dr, dy) \\ &\leq C \int_0^t \int_{|y|>1} (|y| \wedge 1) N(dr, dy), t \in [0, 1]. \end{aligned}$$

Hence

$$\mathbf{E}[|V_t|^\alpha \wedge 1] \leq Ct, t \in [0, 1].$$

Obviously, $|P_t| \leq Ct$, $t \in [0, 1]$. Hence (iii) holds. \square

A straightforward consequence of Lemma 6 is the following statement.

Corollary 3. Let $\alpha \in [1, 2)$, $0 \leq \rho(y) \leq K$, $|b| \leq K$, $\|G\| \leq K$.

(i) For each $p \in (0, \alpha)$, there is $C = C(\alpha, K, d, p)$ such that for all $t \in [0, 1]$,

$$\mathbf{E}\left[\left|X_t^n - X_{\pi_n(t)}^n\right|^p\right] \leq Cn^{-p/\alpha} \text{ if } \alpha \in (1, 2),$$

and

$$\mathbf{E}\left[\left|X_t^n - X_{\pi_n(t)}^n\right|^p\right] \leq C(n/\ln n)^{-p} \text{ if } \alpha = 1.$$

(ii) If $\alpha = 1$, $\rho(y) = \rho(-y)$, $y \in \mathbf{R}^d$. Then for each $p \in (0, \alpha)$ there is $C = C(p, d, K)$ such that for all $t \in [0, 1]$

$$\mathbf{E}\left[\left|X_t^n - X_{\pi_n(t)}^n\right|^p\right] \leq Cn^{-p}.$$

(iii) There is $C = C(\alpha, K, d)$ such that for all $t \in [0, 1]$,

$$\mathbf{E}\left[\left|X_t^n - X_{\pi_n(t)}^n\right|^\alpha \wedge 1\right] \leq C(n/\ln n)^{-1}.$$

Proof. For $\forall t \in [0, 1)$, there is $j \in \{0, 1, \dots, n-1\}$ so that $j/n \leq t < (j+1)/n$, and $\pi_n(t) := j/n$. Thus $0 \leq t - \pi_n(t) \leq 1/n$. Note that for any $S, t > 0$, $L_t = L_{S+t} - L_S$ in distribution. All the estimates immediately follow from Lemma 6. \square

Finally, applying Lemma 5 we derive

Corollary 4. *Let $0 \leq \rho(y) \leq K$.*

(i) There is $C = C(\alpha, d, p, K)$ such that for all $t \in [0, 1)$,

$$\mathbf{E} \left[\left| Y_t^n - Y_{\pi_n(t)}^n \right|^p \right] \leq C \begin{cases} n^{-1} & \text{if } p > \alpha \in [1, 2), \\ (n/\ln n)^{-1} & \text{if } p = \alpha \in [1, 2), \\ n^{-p/\alpha} & \text{if } p < \alpha \in (1, 2), \end{cases}$$

and

$$\mathbf{E} \left[\left| Y_t^n - Y_{\pi_n(t)}^n \right|^p \right] \leq C (n/\ln n)^{-p} \text{ if } p < \alpha = 1.$$

(ii) Let $\alpha = 1$, $\rho(y) = \rho(-y)$, $y \in \mathbf{R}^d$. There is $C = C(p, d, K)$ such that for all $t \in [0, 1)$,

$$\mathbf{E} \left[\left| Y_t^n - Y_{\pi_n(t)}^n \right|^p \right] \leq C n^{-p} \text{ if } p < \alpha = 1.$$

Proof. For $\forall t \in [0, 1)$, there is $j \in \{0, 1, \dots, n-1\}$ so that $j/n \leq t < (j+1)/n$, and $\pi_n(t) := j/n$. Thus $0 \leq t - \pi_n(t) \leq 1/n$. Note that for any $S, t > 0$, $L_t^0 = L_{S+t}^0 - L_S^0$ in distribution. All the estimates immediately follow from Lemma 5. \square

3. PROOF OF MAIN RESULTS

We start with the Lipschitz, possibly completely degenerate, case and derive the rate of convergence directly.

3.1. Proof of Proposition 3. Note that

$$L_t = L_t^0 + V_t - 1_{\alpha \in (1, 2)} t \int_{|y| > 1} y \rho(y) \frac{dy}{|y|^{d+\alpha}}, t \in [0, 1],$$

where

$$V_t = \int_0^t \int_{|y| > 1} y N(dr, dy), t \in [0, 1].$$

Denote

$$\tilde{b}(x) = b(x) - 1_{\alpha \in (1, 2)} G(x) \int_{|y| > 1} y \rho(y) \frac{dy}{|y|^{d+\alpha}}, x \in \mathbf{R}^d.$$

Let X_t be the strong solution to (1.1) and $\bar{X}_t^n := X_t^n - X_t, t \in [0, 1]$. Let $0 \leq S \leq T \leq 1$. Then

$$\begin{aligned} \bar{X}_t^n &= \bar{X}_S^n + \int_S^t [\tilde{b}(X_{\pi_n(r)}^n) - \tilde{b}(X_r^n)]dr + \int_S^t [\tilde{b}(X_r^n) - \tilde{b}(X_r)]dr \\ &\quad + \int_S^t [G(X_{\pi_n(r)}^n) - G(X_{r-}^n)]dL_r^0 + \int_S^t [G(X_{r-}^n) - G(X_{r-})]dL_r^0 \\ &\quad + \int_S^t [G(X_{\pi_n(r)}^n) - G(X_{r-}^n)]dV_r + \int_S^t [G(X_{r-}^n) - G(X_{r-})]dV_r \\ &:= \bar{X}_S^n + \sum_{k=1}^6 A_t^k, t \in [S, T]. \end{aligned}$$

Estimates of A_t^1 and A_t^2 . For $p \in [1, \alpha), \alpha \in (1, 2)$, by Hölder inequality,

$$\begin{aligned} \mathbf{E} \left[\sup_{S \leq t \leq T} |A_t^1 + A_t^2|^p \right] &\leq C \mathbf{E} \left[\left(\int_S^T [|X_{\pi_n(r)}^n - X_r^n| \wedge 1]dr \right)^p + \left(\int_S^T |\bar{X}_r^n|dr \right)^p \right] \\ &\leq C \mathbf{E} \left[\int_S^T [|X_{\pi_n(r)}^n - X_r^n|^p \wedge 1]dr + (T-S)^p \sup_{S \leq t \leq T} |\bar{X}_t^n|^p \right], \end{aligned}$$

and for $p \in (0, 1), \alpha \in [1, 2)$,

$$\begin{aligned} \mathbf{E} \left[\sup_{S \leq t \leq T} |A_t^1 + A_t^2|^p \right] &\leq C \left(\mathbf{E} \int_S^T [|X_{\pi_n(r)}^n - X_r^n| \wedge 1]dr \right)^p \\ &\quad + C(T-S)^p \mathbf{E} \left[\sup_{S \leq t \leq T} |\bar{X}_t^n|^p \right] \end{aligned}$$

for some $C = C(\alpha, d, K, p)$. By Corollary 3, for $p \in (0, \alpha)$ there is $C = C(\alpha, d, K, p)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |A_t^1 + A_t^2|^p \right] \leq C[l(n)^{-p/\alpha} + (T-S)^p \mathbf{E}[\sup_{S \leq t \leq T} |\bar{X}_t^n|^p],$$

where $l(n) = n$ if $p \in (0, \alpha), \alpha \in (1, 2)$, and $l(n) = n/\ln n$ if $0 < p < \alpha = 1$.

Estimate of A_t^3 . By definition,

$$A_t^3 = \int_S^t \int_{|y| \leq 1} [G(X_{\pi_n(r)}^n) - G(X_{r-}^n)] y q(dr, dy), t \in [S, T].$$

According to Corollary 3, there is $C = C(\alpha, d, K)$ so that

$$R := \mathbf{E} \int_S^T [|X_{\pi_n(r)}^n - X_r^n|^\alpha \wedge 1]dr \leq C(n/\ln n)^{-1}.$$

Apply Lemma 2 with $\bar{F}_r = \left| G\left(X_{\pi_n(r)}^n\right) - G\left(X_{r-}^n\right) \right|, r \in [0, 1]$, then for all $p \in (0, \alpha), \alpha \in (1, 2)$, there is $C = C(\alpha, d, K)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |A_t^3|^p \right] \leq C R^{p/\alpha} = C (n/\ln n)^{-p/\alpha}.$$

If $\alpha = 1, p \in (0, 1)$, by Lemma 2 and Hölder inequality,

$$\begin{aligned} \mathbf{E} \left[\sup_{S \leq t \leq T} |A_t^3|^p \right] &\leq C R^p (1 + |\ln R|)^p \leq C (n/\ln n)^{-p} [1 + \ln(n/\ln n)]^p \\ &\leq C [n/(\ln n)^2]^{-p}. \end{aligned}$$

If $\alpha = 1, p \in (0, 1)$, and $\rho(y) = \rho(-y), y \in \mathbf{R}^d$, then applying Lemma 3 with $H_r = G\left(X_{\pi_n(r)}^n\right) - G\left(X_{r-}^n\right), M = 0$, we have

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |A_t^3|^p \right] \leq C (n/\ln n)^{-p}.$$

Estimate of A_t^4 . By definition,

$$A_t^4 = \int_S^t \int_{|y| \leq 1} [G(X_{r-}^n) - G(X_r^n)] y q(dr, dy), t \in [S, T].$$

According to Remark 1, for $p \in (0, 2)$, there is $C = C(p, K)$ so that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |A_t^4|^p \right] \leq C \mathbf{E} \left[\left(\int_S^T |\bar{X}_r^n|^2 dr \right)^{p/2} \right] \leq C (T - S)^{p/2} \mathbf{E} \left[\sup_{S \leq t \leq T} |\bar{X}_t^n|^p \right].$$

Estimate of A_t^5 . By definition,

$$A_t^5 = \int_S^t \int_{|y| > 1} [G(X_{\pi_n(r)}^n) - G(X_{r-}^n)] y N(dr, dy), t \in [S, T].$$

Applying Lemma 4 with $F_r(y) = [G(X_{\pi_n(r)}^n) - G(X_{r-}^n)]y, \bar{F}_r = |G(X_{\pi_n(r)}^n) - G(X_{r-}^n)|, r \in [0, 1], y \in \mathbf{R}^d$, and combining

$$\bar{F}_r = 2K \left(|X_{\pi_n(r)}^n - X_r^n| \wedge 1 \right), r \in [0, 1],$$

we can conclude for $p \in (0, \alpha)$ that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |A_t^5|^p \right] \leq C \mathbf{E} \int_S^T (|X_{\pi_n(r)}^n - X_r^n|^p \wedge 1) dr.$$

Hence by Corollary 3,

$$\begin{aligned}\mathbf{E} \left[\sup_{S \leq t \leq T} |A_t^5|^p \right] &\leq C (n/\ln n)^{-p} \text{ if } p < \alpha = 1, \\ \mathbf{E} \left[\sup_{S \leq t \leq T} |A_t^5|^p \right] &\leq C n^{-p/\alpha} \text{ if } 0 < p < \alpha \in (1, 2).\end{aligned}$$

Estimate of A_t^6 . By definition,

$$A_t^6 = \int_S^t \int_{|y|>1} [G(X_{r-}^n) - G(X_{r-})] y N(dr, dy), t \in [S, T].$$

By Lemma 10 (ii), for $p \in [1, \alpha], \alpha \in (1, 2)$ there is $C = C(\alpha, d, p, K)$ so that

$$\begin{aligned}\mathbf{E} \left[\sup_{S \leq t \leq T} |A_t^6|^p \right] &\leq C \mathbf{E} \left[\left(\int_S^T |\bar{X}_r^n| dr \right)^p + \int_S^T |\bar{X}_r^n|^p dr \right] \\ &\leq C \mathbf{E} \int_S^T |\bar{X}_r^n|^p dr \leq C(T-S) \mathbf{E} \left[\sup_{S \leq t \leq T} |\bar{X}_t^n|^p \right].\end{aligned}$$

According to Remark 3, for $p \in (0, 1)$ there is $C = C(\alpha, d, p, K)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |A_t^6|^p \right] \leq C(T-S) \mathbf{E} \left[\sup_{S \leq t \leq T} |\bar{X}_t^n|^p \right].$$

Summarizing, for $p \in (0, \alpha)$ there is $C = C(\alpha, d, p, K)$ so that for any $S \leq t \leq T \leq 1$,

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |\bar{X}_t^n|^p \right] \leq C \left\{ (n/\ln n)^{-p/\alpha} + (T-S)^{p/2} \mathbf{E} \left[\sup_{S \leq t \leq T} |\bar{X}_t^n|^p \right] \right\}$$

if $\alpha \in (1, 2)$, and

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |\bar{X}_t^n|^p \right] \leq C \left\{ [n/(\ln n)^2]^{-p} + (T-S)^{p/2} \mathbf{E} \left[\sup_{S \leq t \leq T} |\bar{X}_t^n|^p \right] \right\}$$

if $\alpha = 1$. If $\alpha = 1$, and $\rho(y) = \rho(-y), y \in \mathbf{R}^d$, then

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |\bar{X}_t^n|^p \right] \leq C \left\{ (n/\ln n)^{-p} + (T-S)^{p/2} \mathbf{E} \left[\sup_{S \leq t \leq T} |\bar{X}_t^n|^p \right] \right\}.$$

If $C(T-S)^{p/2} \leq 1/2$, then there is $\tilde{C} = \tilde{C}(\alpha, d, p, K)$ such that for $p \in (0, \alpha)$,

$$(3.1) \quad \begin{aligned} \mathbf{E} \left[\sup_{S \leq t \leq T} |\bar{X}_t^n|^p \right] &\leq \tilde{C} (n/\ln n)^{-p/\alpha} \text{ if } \alpha \in (1, 2), \\ \mathbf{E} \left[\sup_{S \leq t \leq T} |\bar{X}_t^n|^p \right] &\leq \tilde{C} [n/(\ln n)^2]^{-p} \text{ if } \alpha = 1, \\ \mathbf{E} \left[\sup_{S \leq t \leq T} |\bar{X}_t^n|^p \right] &\leq \tilde{C} (n/\ln n)^{-p} \text{ if } \alpha = 1 \text{ with symmetry.} \end{aligned}$$

The claim now follows by Lemma 11.

3.2. Proof of Proposition 4. Let Y_t be the strong solution to (1.4) and $\bar{Y}_t^n = Y_t^n - Y_t, t \in [0, 1]$. Let $0 \leq S \leq T \leq 1$. Then

$$\begin{aligned} \bar{Y}_t^n &= \bar{Y}_S^n + \int_S^t [b(Y_{\pi_n(r)}^n) - b(Y_r^n)] dr + \int_S^t [b(Y_r^n) - b(Y_r)] dr \\ &\quad + \int_S^t [G(Y_{\pi_n(r)}^n) - G(Y_{r-}^n)] dL_r^0 + \int_S^t [G(Y_{r-}^n) - G(Y_{r-})] dL_r^0 \\ &:= \bar{Y}_S^n + B_t^1 + B_t^2 + B_t^3 + B_t^4, t \in [S, T]. \end{aligned}$$

Estimates for $p \in (0, \alpha)$ of $B^k, k = 1, \dots, 4$, are identical to estimates of $A^k, k = 1, \dots, 4$, and the conclusion (3.1) holds for $p \in (0, \alpha)$ with \bar{X}^n replaced by \bar{Y}^n .

Estimates of B_t^1 and B_t^2 for $p \in [\alpha, \infty)$. By Hölder inequality and Corollary 4,

$$\begin{aligned} \mathbf{E} \left[\sup_{S \leq t \leq T} |B_t^1|^\alpha \right] &\leq C \mathbf{E} \int_S^T |Y_{\pi_n(r)}^n - Y_r^n|^\alpha dr \leq C (n/\ln n)^{-1}, \\ \mathbf{E} \left[\sup_{S \leq t \leq T} |B_t^1|^p \right] &\leq C \mathbf{E} \int_S^T |Y_{\pi_n(r)}^n - Y_r^n|^p dr \leq C n^{-1} \text{ if } p > \alpha. \end{aligned}$$

By Hölder inequality, for $p \in [\alpha, \infty)$ there is $C = C(K)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |B_t^2|^p \right] \leq C \mathbf{E} \int_S^T |\bar{Y}_r^n|^p dr \leq C(T-S) \mathbf{E} \left[\sup_{S \leq t \leq T} |\bar{Y}_t^n|^p \right].$$

Estimate of B_t^3 for $p \in [\alpha, \infty)$. By definition,

$$B_t^3 = \int_S^t \int_{|y| \leq 1} [G(Y_{\pi_n(r)}^n) - G(Y_{r-}^n)] y q(dr, dy), t \in [S, T].$$

By Corollary 4, there is $C = C(\alpha, d, K)$ so that

$$R := \mathbf{E} \int_S^T |Y_{\pi_n(r)}^n - Y_r^n|^\alpha dr \leq C (n/\ln n)^{-1}.$$

Applying Lemma 2(ii) with $\bar{F}_r = \left| G(Y_{\pi_n(r)}^n) - G(Y_{r-}^n) \right|, r \in [0, 1]$, we can claim there is $C = C(\alpha, d, K)$ such that

$$\begin{aligned} \mathbf{E} \left[\sup_{S \leq t \leq T} |B_t^3|^p \right] &\leq CR(1 + \ln R) \leq C(n/\ln n)^{-1} [1 + \ln(n/\ln n)] \\ &\leq C \left[n/(\ln n)^2 \right]^{-1}. \end{aligned}$$

Now, for $p > \alpha$, by Lemma 1 and Corollary 4, there is $C = C(d, p, \alpha, K)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |B_t^3|^p \right] \leq C \mathbf{E} \int_S^T |\bar{Y}_r^n|^p dr \leq Cn^{-1}.$$

Estimate of B_t^4 for $p \in [\alpha, \infty)$. By definition,

$$B_t^4 = \int_S^t \int_{|y| \leq 1} [G(Y_{r-}^n) - G(Y_{r-})] y q(dr, dy), t \in [S, T].$$

By Lemma 10(i) (Kunita's inequality) and Remark 1, there is $C = C(\alpha, d, p, K)$ such that

$$\begin{aligned} \mathbf{E} \left[\sup_{S \leq t \leq T} |B_t^4|^p \right] &\leq C \mathbf{E} \left[\left(\int_S^T |\bar{Y}_r^n|^2 dr \right)^{p/2} + \int_S^T |\bar{Y}_r^n|^p dr \right] \\ &\leq C \left[(T - S) + (T - S)^{p/2} \right] \mathbf{E} \left[\sup_{S \leq t \leq T} |\bar{Y}_t^n|^p \right]. \end{aligned}$$

Summarizing, there is $C = C(\alpha, d, p, K)$ so that for any $S \leq t \leq T \leq 1$,

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |\bar{Y}_t^n|^\alpha \right] \leq C \left\{ \left[n/(\ln n)^2 \right]^{-1} + (T - S)^{\alpha/2} \mathbf{E} \left[\sup_{S \leq t \leq T} |\bar{Y}_t^n|^\alpha \right] \right\},$$

and for all $p > \alpha$

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |\bar{X}_t^n|^p \right] \leq C \left\{ n^{-1} + (T - S)^{\alpha/2} \mathbf{E} \left[\sup_{S \leq t \leq T} |\bar{Y}_t^n|^p \right] \right\}.$$

We finish the proof by taking $C(T - S) \leq 1/2$ and applying Lemma 11.

3.3. Proof of Proposition 1. First we prove that the Euler approximation sequence is a Cauchy sequence.

Lemma 7. *Let $\alpha \in [1, 2)$, $\beta \in (0, 1)$, $\beta > 1 - \alpha/2$, $p \in (0, \alpha)$ and $\mathbf{S}(c_0), \mathbf{A}(K, c_0)$ hold. Assume, without loss of generality, $|\rho|_\beta \leq K$, $|b|_\beta \leq K$ for the same K . Then there are constants $C_1 = C_1(\alpha, \beta, d, K, c_0, p)$, $c_1 = c_1(\alpha, \beta, d, K, c_0, p)$ such that for any $0 \leq S \leq T \leq 1$ with $T - S \leq c_1$ we have*

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |X_t^n - X_t^m|^p \right] \leq C_1 \left(\mathbf{E} [|X_S^n - X_S^m|^p] + n^{-p\beta/\alpha} + m^{-p\beta/\alpha} \right).$$

Moreover, if X_t is a strong solution to (1.1), then

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |X_t^n - X_t|^p \right] \leq C_1 \left(\mathbf{E} [|X_S^n - X_S|^p] + n^{-p\beta/\alpha} \right).$$

Proof. By Corollary 2, for each $k = 1, \dots, d$, there exists a unique solution $u^k(t, x)$ to (2.4) with

$$\tilde{b}(x) = b(x) - 1_{\alpha \in (1,2)} G(x) \int_{|y|>1} y \rho(y) \frac{dy}{|y|^{d+\alpha}}, x \in \mathbf{R}^d.$$

Note that \tilde{b} is also a bounded β -Hölder continuous function. Denote $u = (u^k)_{1 \leq k \leq d}$. By Itô formula and definition of Euler approximation (1.4), for $t \in [S, T]$, using (2.4),

$$\begin{aligned} & u^k(t, X_t^n) - u^k(S, X_S^n) \\ &= \int_S^t \tilde{b}^k(X_r^n) dr + \int_S^t \left[\tilde{b}^k(X_{\pi_n(r)}^n) - \tilde{b}^k(X_r^n) \right] \cdot \nabla u^k(r, X_r^n) dr \\ &+ \int_S^t \int_{|y| \leq 1} \left[u^k(r, X_{r-}^n + G(X_{\pi_n(r)}^n) y) - u^k(r, X_{r-}^n) \right] q(dr, dy) \\ &+ \int_S^t \int_{|y| > 1} \left[u^k(r, X_{r-}^n + G(X_{\pi_n(r)}^n) y) - u^k(r, X_{r-}^n) \right] N(dr, dy) \\ &+ \int_S^t \int_{|y| \leq 1} \{ u^k(r, X_r^n + G(X_{\pi_n(r)}^n) y) - u^k(r, X_r^n + G(X_r^n) y) \\ &\quad - \nabla u^k(r, X_r^n) \cdot [G(X_{\pi_n(r)}^n) - G(X_{r-}^n)] y \} \rho(y) \frac{dy dr}{|y|^{d+\alpha}}. \end{aligned}$$

On the other hand, according to (1.3), for $t \in [S, T]$

$$\begin{aligned} & X_t^n - X_S^n \\ &= \int_S^t \tilde{b}(X_r^n) dr + \int_S^t \left[\tilde{b}(X_{\pi_n(r)}^n) - \tilde{b}(X_r^n) \right] dr \\ &+ \int_S^t \int_{|y| > 1} G(X_{\pi_n(r)}^n) y N(dr, dy) + \int_S^t \int_{|y| \leq 1} G(X_{\pi_n(r)}^n) y q(dr, dy). \end{aligned}$$

It follows from the two identities above that

$$X_t^n = \sum_{k=1}^7 D_t^{n,k},$$

where

$$\begin{aligned}
D_t^{n,1} &= X_S^n + [u(t, X_t^n) - u(S, X_S^n)], \\
D_t^{n,2} &= \int_S^t [\tilde{b}(X_{\pi_n(r)}^n) - \tilde{b}(X_r^n)] (I_d - \nabla u(r, X_r^n)) dr, \\
D_t^{n,3} &= \int_S^t \int_{|y| \leq 1} \{u(r, X_r^n + G(X_r^n)y) - u(r, X_r^n + G(X_{\pi_n(r)}^n)y) \\
&\quad - \nabla u(r, X_r^n) \cdot [G(X_r^n) - G(X_{\pi_n(r)}^n)]y\} \rho(y) \frac{dy dr}{|y|^{d+\alpha}}, \\
D_t^{n,4} &= \int_S^t \int_{|y| \leq 1} \{[u(r, X_{r-}^n + G(X_{r-}^n)y) - u(r, X_{r-}^n + G(X_{\pi_n(r)}^n)y)] \\
&\quad + [G(X_{\pi_n(r)}^n) - G(X_{r-}^n)]y\} q(dr, dy), \\
D_t^{n,5} &= \int_S^t \int_{|y| > 1} [u(r, X_{r-}^n + G(X_{r-}^n)y) - u(r, X_{r-}^n + G(X_{\pi_n(r)}^n)y)] \\
&\quad + [G(X_{\pi_n(r)}^n) - G(X_{r-}^n)]y\} N(dr, dy), \\
D_t^{n,6} &= \int_S^t \int_{|y| \leq 1} \{G(X_{r-}^n)y - [u(r, X_{r-}^n + G(X_{r-}^n)y) - u(r, X_{r-}^n)]\} q(dr, dy), \\
D_t^{n,7} &= \int_S^t \int_{|y| > 1} \{G(X_{r-}^n)y - [u(r, X_{r-}^n + G(X_{r-}^n)y) - u(r, X_{r-}^n)]\} N(dr, dy).
\end{aligned}$$

Let $D_t^{n,m;k} = D_t^{n,k} - D_t^{m,k}$, and $X_t^{n,m} = X_t^n - X_t^m$, $n, m \geq 1, k = 1, \dots, 7$.

Estimate of $D_t^{n,m;1}$. Using the terminal condition of (2.4) and Corollary 2, we see that for $p \in (0, \infty)$ there is a constant $C = C(\alpha, \beta, d, K, p, c_0)$ such that

$$\begin{aligned}
|D_t^{n,m;1}|^p &\leq C\{|X_S^m - X_S^n|^p + |u(t, X_t^n) - u(t, X_t^m) + u(T, X_t^m) - u(T, X_t^n)|^p \\
&\quad + |u(T, X_S^n) - u(T, X_S^m) + u(S, X_S^m) - u(S, X_S^n)|^p\} \\
&\leq C\{|X_S^n - X_S^m|^p + (T-t)^{p/2} |X_t^n - X_t^m|^p\},
\end{aligned}$$

therefore,

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |D_t^{n,m;1}|^p \right] \leq C\{(T-S)^{p/2} \mathbf{E} \left[\sup_{S \leq t \leq T} |X_t^{n,m}|^p \right] + \mathbf{E} |X_S^{n,m}|^p\}.$$

Estimate of $D_t^{n,m;2}$. Obviously, $|D_t^{n,m;2}|^p \leq 2^p[|D_t^{n,2}|^p + |D_t^{m,2}|^p]$, $t \in [0, 1]$.

For $p \in [1, \alpha)$, $\alpha \in (1, 2)$, by Hölder inequality and Corollary 3, there is $C = C(\alpha, \beta, d, c_0, K, p)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |D_t^{n,2}|^p \right] \leq C \mathbf{E} \left[\int_S^T |X_{\pi_n(r)}^n - X_r^n|^{p\beta} dr \right] \leq C n^{-\beta p/\alpha}.$$

For $p \in (0, 1)$, by Corollary 3, there is a constant $C = C(\alpha, \beta, d, K, p)$ such that

$$\begin{aligned} \mathbf{E} \left[\sup_{S \leq t \leq T} |D_t^{n,2}|^p \right] &\leq C \mathbf{E} \left[\left(\int_S^T |X_{\pi_n(r)}^n - X_r^n|^\beta dr \right)^p \right] \\ &\leq C \left(\int_S^T \mathbf{E}[|X_{\pi_n(r)}^n - X_r^n|^\beta] dr \right)^p \leq C n^{-\beta p/\alpha}. \end{aligned}$$

Similarly, we can obtain the estimates for $D_t^{m,2}$. Hence, by Hölder inequality, for all $p \in (0, \alpha)$, there is $C = C(\alpha, \beta, d, K, p)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |D_t^{n,m;2}|^p \right] \leq C[n^{-p\beta/\alpha} + m^{-p\beta/\alpha}].$$

Estimate of $D_t^{n,m;3}$. Obviously, $|D_t^{n,m;3}|^p \leq 2^p[|D_t^{n,3}|^p + |D_t^{m,3}|^p]$, $t \in [0, 1)$. Note that

$$\begin{aligned} |D_t^{n,3}| &= \left| \int_S^t \int_{|y| \leq 1} \left\{ \left[\int_0^1 [\nabla u(r, X_r^n) - \nabla u(r, X_r^n + G(X_r^n)y + s[G(X_{\pi_n(r)}^n) - G(X_r^n)]y)] ds \cdot [G(X_{\pi_n(r)}^n) - G(X_r^n)]y \right] \rho(y) \frac{dy dr}{|y|^{d+\alpha}} \right\} \right|. \end{aligned}$$

Let $\beta' \in (0, 1)$, $\alpha + \beta > 1 + \beta' > \alpha$ and denote $G_r^n = |G(X_{\pi_n(r)}^n) - G(X_r^n)|$, $r \in [0, 1)$. Then there is $C = C(\alpha, d, K, c_0, \beta)$ such that

$$\begin{aligned} |D_t^{n,3}| &\leq C \int_S^T \int_{|y| \leq 1} G_r^n |y|^{1+\beta'} \frac{dy}{|y|^{d+\alpha}} dr \\ &\leq C \int_S^T |X_r^n - X_{\pi_n(r)}^n| \wedge 1 dr, \quad t \in [S, T]. \end{aligned}$$

Hence by Corollary 3, for $p \in [1, \alpha)$, $\alpha \in (1, 2)$,

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |D_t^{n,3}|^p \right] \leq C \int_S^T \mathbf{E} [|X_r^n - X_{\pi_n(r)}^n|^p] dr \leq C n^{-p/\alpha},$$

for $p \in (0, 1)$, according to Corollary 3,

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |D_t^{n,3}|^p \right] \leq C \left(\int_S^T \mathbf{E} [|X_r^n - X_{\pi_n(r)}^n| \wedge 1] dr \right)^p \leq C (n/\ln n)^{-p/\alpha}.$$

Similar reasoning can be applied to $|D_t^{m,3}|$. Therefore for all $p \in (0, \alpha)$ there is $C = C(\alpha, d, p, K, \beta)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |D_t^{n,m;3}|^p \right] \leq C [(n/\ln n)^{-p/\alpha} + (m/\ln m)^{-p/\alpha}].$$

Estimate of $D_t^{n,m;4}$. Obviously, $\left|D_t^{n,m;4}\right|^p \leq 2^p[\left|D_t^{n,4}\right|^p + \left|D_t^{m,4}\right|^p], t \in [0, 1)$. By Corollary 3(iii), there is $C = C(\alpha, d, K)$ such that

$$R := \mathbf{E} \int_S^T \left|X_{\pi_n(r)}^n - X_{r-}^n\right|^\alpha \wedge 1 dr \leq C (n/\ln n)^{-1}.$$

First, let $\alpha \in (1, 2)$. Applying Lemma 2(i) to $D_t^{n,4}$ with

$$(3.2) \quad \bar{F}_r = 2K(1 + |\nabla u|_0) \left(\left|X_{\pi_n(r)}^n - X_{r-}^n\right| \wedge 1 \right), r \in [S, T],$$

and Corollary 3(iii), we have that for $p \in (0, \alpha)$ there is $C = C(\alpha, d, p, K, \beta, c_0)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} \left|D_t^{n,4}\right|^p \right] \leq CR^{p/\alpha} \leq C (n/\ln n)^{-p/\alpha}.$$

Now, let $\alpha = 1$. Applying Lemma 3(i) to $D_t^{n,4}$ with \bar{F}_r given by (3.2), and Corollary 3(iii), we see there is $C = C(\alpha, d, p, K, \beta)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} \left|D_t^{n,4}\right|^p \right] \leq CR^p (1 + |\ln R|)^p \leq C \left[n/(\ln n)^2 \right]^{-p}.$$

Similarly,

$$\mathbf{E} \left[\sup_{S \leq t \leq T} \left|D_t^{m,4}\right|^p \right] \leq C \left[m/(\ln m)^2 \right]^{-p/\alpha},$$

and thus there is $C = C(\alpha, d, p, K, \beta, c_0)$ so that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} \left|D_t^{n,m;4}\right|^p \right] \leq C \left\{ \left[n/(\ln n)^2 \right]^{-p/\alpha} + \left[m/(\ln m)^2 \right]^{-p/\alpha} \right\}.$$

Estimate of $D_t^{n,m;5}$. Obviously, $\left|D_t^{n,m;5}\right|^p \leq 2^p[\left|D_t^{n,5}\right|^p + \left|D_t^{m,5}\right|^p], t \in [0, 1)$. By Lemma 4, applied to $D_t^{n,5}$ with

$$\bar{F}_r = (1 + |\nabla u|_0) |\nabla G|_\infty \left|X_{\pi_n(r)}^n - X_{r-}^n\right|, r \in [S, T],$$

and Corollary 3, there is $C = C(\alpha, d, p, K, \beta, c_0)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} \left|D_t^{n,5}\right|^p \right] \leq C n^{-p/\alpha} \text{ for } p \in (0, \alpha).$$

Similarly as above, for $p \in (0, \alpha)$,

$$\mathbf{E} \left[\sup_{S \leq t \leq T} \left|D_t^{n,m;5}\right|^p \right] \leq C \left[n^{-p/\alpha} + m^{-p/\alpha} \right].$$

Estimate of $D_t^{n,m;6}$. Denote $G_r^{n,m} = G(X_r^n) - G(X_r^m)$, $r \in [S, T]$. Then

$$\begin{aligned}
D_t^{n,m;6} &= \int_S^t \int_{|y| \leq 1} \left\{ [G(X_{r-}^n) - G(X_{r-}^m)] y \right. \\
&\quad \left. - [u(r, X_{r-}^n + G(X_{r-}^n) y) - u(r, X_{r-}^n + G(X_{r-}^m) y)] \right\} q(dr, dy) \\
&\quad - \int_S^t \int_{|y| \leq 1} \{ [u(r, X_{r-}^n + G(X_{r-}^m) y) - u(r, X_{r-}^m + G(X_{r-}^m) y)] \\
&\quad + [u(r, X_r^m) - u(r, X_r^n)] \} q(dr, dy) \\
&:= D_t^{n,m;61} + D_t^{n,m;62}.
\end{aligned}$$

For $p \in (0, 2)$, by Remark 1, there is $C = C(\alpha, \beta, d, K, c_0, p)$ such that

$$\begin{aligned}
\mathbf{E} \left[\sup_{S \leq t \leq T} |D_t^{n,m;61}|^p \right] &\leq C \mathbf{E} \left[\left(\int_S^T \int_{|y| \leq 1} |G_r^{n,m} y|^2 \frac{dy dr}{|y|^{d+\alpha}} \right)^{p/2} \right] \\
&\leq C (T - S)^{p/2} \mathbf{E} \left[\sup_{S \leq r \leq T} |X_r^{n,m}|^p \right].
\end{aligned}$$

We rewrite

$$\begin{aligned}
D_t^{n,m;62} &= \int_S^t \int_{|y| \leq 1} \int_0^1 [-\nabla u(r, X_{r-}^m + G(X_{r-}^m) y + s X_{r-}^{n,m}) \\
&\quad + \nabla u(r, X_{r-}^m + s X_{r-}^{n,m})] X_{r-}^{n,m} ds q(dr, dy), t \in [S, T].
\end{aligned}$$

Let $1 + \beta' < \alpha + \beta$ and $2\beta' > \alpha$. Then by Remark 1, there is $C = C(\alpha, \beta, K, p, c_0, d)$ such that

$$\begin{aligned}
\mathbf{E} \left[\sup_{S \leq t \leq T} |D_t^{n,m;62}|^p \right] &\leq C \mathbf{E} \left[\left(\int_S^T |X_r^{n,m}|^2 dr \int_{|y| \leq 1} |y|^{2\beta'} \frac{dy}{|y|^{d+\alpha}} \right)^{p/2} \right] \\
&\leq C (T - S)^{p/2} \mathbf{E} \left[\sup_{S \leq t \leq T} |X_r^{n,m}|^p \right].
\end{aligned}$$

Hence,

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |D_t^{n,m;6}|^p \right] \leq C (T - S)^{p/2} \mathbf{E} \left[\sup_{S \leq t \leq T} |X_r^{n,m}|^p \right].$$

Estimate of $D_t^{n,m;7}$. Let $\alpha \in (1, 2)$, $p \in [1, \alpha]$. By Lemma 10(ii) (see Lemma 4.1 in [6]), there is $C = C(\alpha, \beta, K, c_0, p, d)$ such that

$$\begin{aligned} \mathbf{E} \left[\sup_{S \leq t \leq T} |D_t^{n,m;7}|^p \right] &\leq C \mathbf{E} \left[\left(\int_S^T \int_{|y|>1} [|G_r^{n,m} y| + |X_r^{n,m}|] \frac{dy dr}{|y|^{d+\alpha}} \right)^p \right. \\ &\quad \left. + \int_S^T \int_{|y|>1} [|G_r^{n,m} y|^p + |X_r^{n,m}|^p] \frac{dy dr}{|y|^{d+\alpha}} \right] \\ &\leq C (T - S) \mathbf{E} \left[\sup_{S \leq t \leq T} |X_t^{n,m}|^p \right]. \end{aligned}$$

Let $\alpha = 1$, $p \in (0, 1)$. By Remark 3, there is $C = C(\alpha, \beta, K, p, d)$ such that

$$|D_t^{n,m;7}|^p \leq C \int_S^T \int_{|y|>1} [|G_{r-}^{n,m} y| + |X_{r-}^{n,m}|]^p N(dr, dy), t \in [S, T],$$

and thus

$$\begin{aligned} \mathbf{E} \left[\sup_{S \leq t \leq T} |D_t^{n,m;7}|^p \right] &\leq C \mathbf{E} \left[\int_S^T \int_{|y|>1} [|G_r^{n,m} y|^p + |X_r^{n,m}|^p] \frac{dy dr}{|y|^{d+\alpha}} \right] \\ &\leq C (T - S) \mathbf{E} \left[\sup_{S \leq r \leq T} |X_r^{n,m}|^p \right]. \end{aligned}$$

Collecting all the estimates above we see that for $p \in (0, \alpha)$ there is $C = C(\alpha, \beta, K, p, d)$ such that

$$\begin{aligned} \mathbf{E} \left[\sup_{S \leq t \leq T} |X_t^{n,m}|^p \right] &\leq C \left\{ (T - S)^{p/2} \mathbf{E} \left[\sup_{S \leq t \leq T} |X_t^{n,m}|^p \right] + \mathbf{E} [|X_S^{n,m}|^p] \right. \\ (3.3) \quad &\quad \left. + n^{-p\beta/\alpha} + m^{-p\beta/\alpha} \right\}. \end{aligned}$$

Set $c_1 = (2C)^{-2/p}$, $C_1 = 2C$ with the C in (3.3), we then have

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |X_t^{n,m}|^p \right] \leq C_1 \left\{ \mathbf{E} [|X_S^{n,m}|^p] + n^{-p\beta/\alpha} + m^{-p\beta/\alpha} \right\}$$

if $0 \leq T - S \leq c_1$.

Rate of convergence. Now let us assume X_t is a strong solution to (1.1). We have, by Itô formula and (2.4), for $t \in [S, T]$,

$$\begin{aligned} &u(t, X_t) - u(S, X_S) \\ &= \int_S^t \tilde{b}(X_r) dr + \int_S^t \int_{|y| \leq 1} [u(r, X_{r-} + G(X_{r-})y) - u(r, X_{r-})] q(dr, dy) \\ &+ \int_S^t \int_{|y| > 1} [u(r, X_{r-} + G(X_{r-})y) - u(r, X_{r-})] N(dr, dy), \end{aligned}$$

Hence for $t \in [S, T]$, we obtain

$$\begin{aligned} X_t - X_S &= u(t, X_t) - u(S, X_S) \\ &+ \int_S^t \int_{|y| \leq 1} \{G(X_{r-})y - [u(r, X_{r-} + G(X_{r-})y) - u(r, X_{r-})]\} q(dr, dy) \\ &+ \int_S^t \int_{|y| > 1} \{G(X_{r-})y - [u(r, X_{r-} + G(X_{r-})y) - u(r, X_{r-})]\} N(dr, dy), \end{aligned}$$

and

$$\begin{aligned} &X_t^n - X_t \\ &= \{X_S^n - X_S + [u(t, X_t^n) - u(S, X_S^n)] - [u(t, X_t) - u(S, X_S)]\} \\ &+ \sum_{k=2}^5 D_t^{n,k} + D_t^{n,6} + D_t^{n,7} \\ &- \int_S^t \int_{|y| \leq 1} \{G(X_{r-})y - [u(r, X_{r-} + G(X_{r-})y) - u(r, X_{r-})]\} q(dr, dy) \\ &- \int_S^t \int_{|y| > 1} \{G(X_{r-})y - [u(r, X_{r-} + G(X_{r-})y) - u(r, X_{r-})]\} N(dr, dy). \end{aligned}$$

Estimates for $D_t^{n,k}, k = 2, \dots, 5$ have been derived above. And we can estimate

$$\begin{aligned} &X_S^n - X_S + [u(t, X_t^n) - u(S, X_S^n)] - u(t, X_t) - u(S, X_S), \\ &D_t^{n,6} - \int_S^t \int_{|y| \leq 1} \{G(X_{r-})y - [u(r, X_{r-} + G(X_{r-})y) - u(r, X_{r-})]\} q(dr, dy), \\ &D_t^{n,7} - \int_S^t \int_{|y| > 1} \{G(X_{r-})y - [u(r, X_{r-} + G(X_{r-})y) - u(r, X_{r-})]\} N(dr, dy) \end{aligned}$$

in exactly the same way as we estimated $D_t^{n,m;1}, D_t^{n,m;6}$ and $D_t^{n,m;7}$ (by replacing X_t^m by X_t in the arguments). We find that there is a constant $C = C(\alpha, \beta, p, K, c_0, d)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |X_t^n - X_t|^p \right] \leq C_1 \left(\mathbf{E} [|X_S^n - X_S|^p] + n^{-p\beta/\alpha} \right).$$

Then the claimed rate of convergence holds because of Lemma 11. \square

Existence of a solution. Let $p \in (0, \alpha)$ and c_1 be the constant in Lemma 7. By Lemmas 11 and 7, there is $C = C(\alpha, \beta, p, K, c_0, d)$ such that for $n, m \geq 1$,

$$\mathbf{E} \left[\sup_{0 \leq t \leq 1} |X_t^n - X_t^m|^p \right] \leq C \left(n^{-p\beta/\alpha} + m^{-p\beta/\alpha} \right),$$

and thus

$$\mathbf{E} \left[\sup_{0 \leq t \leq 1} |X_t^n - X_t^m|^p \right] \rightarrow 0$$

as $n, m \rightarrow \infty$. Therefore there is an adapted càdlàg process X_t such that for all $p \in (0, \alpha)$,

$$\mathbf{E} \left[\sup_{0 \leq t \leq 1} |X_t^n - X_t|^p \right] \rightarrow 0$$

as $n \rightarrow \infty$. Hence X_t solves (1.1). Moreover, by Lemma 7, there is $C = C(\alpha, \beta, d, K, p)$ such that

$$\mathbf{E} \left[\sup_{0 \leq t \leq 1} |X_t^n - X_t|^p \right] \leq C n^{-p\beta/\alpha}.$$

Uniqueness follows from Lemma 7: any strong solution can be approximated by X_t^n .

3.4. Proof of Proposition 2. The proof repeats the steps we took to prove Proposition 1.

Lemma 8. *Let $\alpha \in [1, 2)$, $\beta \in (0, 1)$, $\beta > 1 - \alpha/2$, $p \in (0, \alpha)$ and $\mathcal{S}(c_0), \mathcal{A}(K, c_0)$ hold. Assume (without loss of generality), $|\rho|_\beta \leq K$, $|b|_\beta \leq K$ for the same K . Then there are constants $C_1 = C_1(\alpha, \beta, d, K, c_0, p)$, $c_1 = c_1(\alpha, \beta, d, K, c_0, p)$ such that for any $0 \leq S \leq T \leq 1$ with $T - S \leq c_1$ we have*

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |Y_t^n - Y_t^m|^p \right] \leq C_1 (\mathbf{E} [|Y_S^n - Y_S^m|^p] + l(n, \beta, \alpha, p) + l(m, \beta, \alpha, p)),$$

where $l(k, \beta, \alpha, p) = k^{-p\beta/\alpha}$ if $p\beta < \alpha$, $l(k, \beta, \alpha, p) = (k/\ln k)^{-1}$ if $p\beta = \alpha$, and $l(k, \beta, \alpha, p) = k^{-1}$ if $p\beta > \alpha$.

Moreover, if Y_t is a strong solution to (1.3), then

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |Y_t^n - Y_t|^p \right] \leq C_1 l(n, \beta, \alpha, p).$$

Proof. Let $0 \leq S \leq T \leq 1$. By Corollary 2, for each $k = 1, \dots, d$, there exists a unique solution $u^k(t, x)$ to (2.4) with $\tilde{b}(x) = b(x)$, $x \in \mathbf{R}^d$. Denote $u = (u^k)_{1 \leq k \leq d}$. By Itô formula and definition of Euler approximation (1.5), for $t \in [S, T]$, using (2.4),

$$\begin{aligned} & u^k(t, Y_t^n) - u^k(S, Y_S^n) \\ &= \int_S^t b^k(Y_r^n) dr + \int_S^t \left[b^k(Y_{\pi_n(r)}^n) - b^k(Y_r^n) \right] \cdot \nabla u^k(r, Y_r^n) dr \\ &+ \int_S^t \int_{|y| \leq 1} \left[u^k(r, Y_{r-}^n + G(Y_{\pi_n(r)}^n) y) - u^k(r, Y_{r-}^n) \right] q(dr, dy) \\ &+ \int_S^t \int_{|y| \leq 1} \{ u^k(r, Y_r^n + G(Y_{\pi_n(r)}^n) y) - u^k(r, Y_r^n + G(Y_r^n) y) \\ &\quad - \nabla u^k(r, Y_r^n) \cdot [G(Y_{\pi_n(r)}^n) - G(Y_r^n)] y \} \rho(y) \frac{dy dr}{|y|^{d+\alpha}}. \end{aligned}$$

On the other hand, according to (1.5), for $t \in [S, T]$

$$\begin{aligned} Y_t^n - Y_S^n &= \int_S^t b(Y_r^n) dr + \int_S^t \left[b(Y_{\pi_n(r)}^n) - b(Y_r^n) \right] dr \\ &\quad + \int_S^t \int_{|y| \leq 1} G(Y_{\pi_n(r)}^n) y q(dr, dy). \end{aligned}$$

It follows from the two identities above that

$$Y_t^n = \sum_{k=1}^4 B_t^{n,k} + B_t^{n,5},$$

where

$$\begin{aligned} B_t^{n,1} &= Y_S^n + [u(t, Y_t^n) - u(S, Y_S^n)], \\ B_t^{n,2} &= \int_S^t \left[b(Y_{\pi_n(r)}^n) - b(Y_r^n) \right] (I_d - \nabla u(r, Y_r^n)) dr, \\ B_t^{n,3} &= \int_S^t \int_{|y| \leq 1} \{ u(r, Y_r^n + G(Y_r^n) y) - u(r, Y_r^n + G(Y_{\pi_n(r)}^n) y) \\ &\quad - \nabla u(r, Y_r^n) \cdot [G(Y_r^n) - G(Y_{\pi_n(r)}^n)] y \} \rho(y) \frac{dy dr}{|y|^{d+\alpha}}, \\ B_t^{n,4} &= \int_S^t \int_{|y| \leq 1} \left\{ [u(r, Y_{r-}^n + G(Y_{r-}^n) y) - u(r, Y_{r-}^n + G(Y_{\pi_n(r)}^n) y)] \right. \\ &\quad \left. + [G(Y_{\pi_n(r)}^n) - G(Y_{r-}^n)] y \right\} q(dr, dy), \\ B_t^{n,5} &= \int_S^t \int_{|y| \leq 1} \left\{ G(Y_{r-}^n) y - [u(r, Y_{r-}^n + G(Y_{r-}^n) y) - u(r, Y_{r-}^n)] \right\} q(dr, dy). \end{aligned}$$

Let $B_t^{n,m;k} = B_t^{n,k} - B_t^{m,k}$, and $Y_t^{n,m} = Y_t^n - Y_t^m$, $n, m \geq 1, k = 1, \dots, 5$.

Estimate of $B_t^{n,m;1}$. This estimate is identical to that of $D^{n,m;1}$ in the proof of Lemma 7. Repeating it and applying Corollary 2, we see that for $p \in (0, \infty)$ there is $C = C(\alpha, \beta, p, K, c_0, d)$ so that

$$\left| B_t^{n,m;1} \right|^p \leq C \{ [|Y_S^n - Y_S^m|^p] + (T - t)^{p/2} |Y_t^n - Y_t^m|^p \},$$

and,

$$\mathbf{E} \left[\sup_{S \leq t \leq T} \left| B_t^{n,m;1} \right|^p \right] \leq C \{ (T - S)^{p/2} \mathbf{E} \left[\sup_{S \leq t \leq T} |Y_t^{n,m}|^p \right] + \mathbf{E} |Y_S^{n,m}|^p \}.$$

Estimates of $B_t^{n,m;k}$, $k = 2, 3, 4$, for $p \in (0, \alpha)$ are identical to the estimates of $D_t^{n,m;k}$, $k = 2, 3, 4$. We replace X by Y , and apply Corollary 4 instead of 3. Note that for $p \in (0, \alpha)$ the estimates in Corollary 3 coincide with estimates in Corollary 4. Hence for $p \in (0, \alpha)$ there is $C = C(\alpha, \beta, p, c_0, K)$ such that

for $k = 2, 3, 4$,

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |B_t^{n,m;k}|^p \right] \leq C \left(n^{-p\beta/\alpha} + m^{-p\beta/\alpha} \right).$$

Estimate of $B_t^{n,m;2}$ for $p \in [\alpha, \infty)$. By Hölder inequality and Corollary 2, there is $C = C(\alpha, \beta, d, K, c_0, p)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |B_t^{n,2}|^p \right] \leq C \mathbf{E} \left[\int_S^T |Y_{\pi_n(r)}^n - Y_r^n|^{p\beta} dr \right].$$

Hence, by Corollary 4,

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |B_t^{n,2}|^p \right] \leq Cl(n, \beta, \alpha, p),$$

where $l(n, \beta, \alpha, p) = n^{-p\beta/\alpha}$ if $p\beta < \alpha$, $l(n, \beta, \alpha, p) = (n/\ln n)^{-1}$ if $p\beta = \alpha$, and $l(n, \beta, \alpha, p) = n^{-1}$ if $p\beta > \alpha$. Therefore for $p \in [\alpha, \infty)$,

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |B_t^{n,m;2}|^p \right] \leq C[l(n, \beta, \alpha, p) + l(m, \beta, \alpha, p)].$$

Estimate of $B_t^{n,m;3}$ for $p \in [\alpha, \infty)$. By repeating the argument for $D_t^{n,3}$ in the proof of Proposition 1, we find that there is $C = C(\alpha, \beta, d, K)$ so that

$$|B_t^{n,3}| \leq C \int_S^T |Y_r^n - Y_{\pi_n(r)}^n| dr, t \in [S, T].$$

Hence by Corollary 4, for $p \geq \alpha$ there is $C = C(\alpha, \beta, d, K, c_0, p)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |B_t^{n,3}|^p \right] \leq C \int_S^T \mathbf{E} [|Y_r^n - Y_{\pi_n(r)}^n|^p] dr \leq Cl(n, 1, \alpha, p).$$

Therefore, for $p \in [\alpha, \infty)$ there is $C = C(\alpha, \beta, p, K, c_0, d)$ such that

$$\begin{aligned} \mathbf{E} \left[\sup_{S \leq t \leq T} |B_t^{n,m;3}|^p \right] &\leq C[l(n, 1, \alpha, p) + l(m, 1, \alpha, p)] \\ &\leq C[l(n, \beta, \alpha, p) + l(m, \beta, \alpha, p)]. \end{aligned}$$

Estimate of $B_t^{n,m;4}$ for $p \in [\alpha, \infty)$.

$$\begin{aligned} B_t^{n,4} &= \int_S^t \int_{|y| \leq 1} \left\{ [u(r, Y_{r-}^n + G(Y_{r-}^n)y) - u(r, Y_{r-}^n + G(Y_{\pi_n(r)}^n)y)] \right. \\ &\quad \left. + [G(Y_{\pi_n(r)}^n) - G(Y_{r-}^n)]y \right\} q(dr, dy), \end{aligned}$$

By Corollary 4(i), there is $C = C(\alpha, d, K)$ such that

$$R := \mathbf{E} \int_S^T |Y_{\pi_n(r)}^n - Y_r^n|^\alpha dr \leq C(n/\ln n)^{-1}.$$

Applying Lemma 2(ii) with

$$(3.4) \quad \bar{F}_r = (1 + |\nabla u|_0) |\nabla G|_\infty \left| Y_{\pi_n(r)}^n - Y_{r-}^n \right|, r \in [S, T],$$

we can see there is $C = C(\alpha, \beta, K, c_0, d)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} \left| B_t^{n,4} \right|^\alpha \right] \leq CR(1 + |\ln R|) \leq C \left[n / (\ln n)^2 \right]^{-1}.$$

By Lemma 1 with \bar{F}_r given by (3.4) and Corollary 4, for $p > \alpha$ there is $C = C(\alpha, \beta, p, d, K, c_0)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} \left| B_t^{n,4} \right|^p \right] \leq Cn^{-1}.$$

Hence for $p \geq \alpha$ there is $C = C(\alpha, \beta, p, d, K, c_0)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} \left| B_t^{n,m;4} \right|^p \right] \leq C[l(n, \beta, \alpha, p) + l(m, \beta, \alpha, p)].$$

Estimate of $B_t^{n,m;5}$. As in the case of $D_t^{n,m;6}$ in the proof of Proposition 1, we rewrite

$$\begin{aligned} B_t^{n,m;5} &= \int_S^t \int_{|y| \leq 1} \left\{ [G(Y_{r-}^n) - G(Y_{r-}^m)] y \right. \\ &\quad \left. - [u(r, Y_{r-}^n + G(Y_{r-}^n) y) - u(r, Y_{r-}^m + G(Y_{r-}^m) y)] \right\} q(dr, dy) \\ &\quad - \int_S^t \int_{|y| \leq 1} \{ [u(r, Y_{r-}^n + G(Y_{r-}^m) y) - u(r, Y_{r-}^m + G(Y_{r-}^m) y)] \\ &\quad + [u(r, Y_{r-}^m) - u(r, Y_{r-}^n)] \} q(dr, dy) \\ &:= B_t^{n,m;51} + B_t^{n,m;52}, \end{aligned}$$

and

$$\begin{aligned} B_t^{n,m;52} &= \int_S^t \int_{|y| \leq 1} \int_0^1 [-\nabla u(r, Y_{r-}^m + G(Y_{r-}^m) y + sY_{r-}^{n,m}) \\ &\quad + \nabla u(r, Y_{r-}^m + sY_{r-}^{n,m})] Y_{r-}^{n,m} ds q(dr, dy), t \in [S, T]. \end{aligned}$$

For $p \in (0, 2)$, repeating the estimates of $D^{m,m;6}$ in the proof of Proposition 1, we find that for $p \in (0, 2)$ there is $C = C(\alpha, p, K, c_0, \beta, d)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} \left| B_t^{n,m;5} \right|^p \right] \leq C(T - S)^{p/2} \mathbf{E} \left[\sup_{S \leq t \leq T} |Y_r^{n,m}|^p \right].$$

For $p \geq 2$, by Lemma 10(i), there is $C = C(\alpha, p, K, c_0, \beta, d)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} \left| B_t^{n,m;5} \right|^p \right] \leq C(T - S) \mathbf{E} \left[\sup_{S \leq t \leq T} |Y_r^{n,m}|^p \right].$$

Collecting all the estimates above we see that for $p \in (0, \infty)$ there is $C = C(\alpha, \beta, K, c_0, p, d)$ such that

$$\begin{aligned} \mathbf{E} \left[\sup_{S \leq t \leq T} |Y_t^{n,m}|^p \right] &\leq C \left\{ [(T-S)^{p/2} + (T-S)] \mathbf{E} \left[\sup_{S \leq t \leq T} |Y_t^{n,m}|^p \right] \right. \\ &\quad \left. + \mathbf{E} [|Y_S^{n,m}|^p] + l(n, \beta, \alpha, p) + l(m, \beta, \alpha, p) \right\}. \end{aligned}$$

There is $c_1 = c_1(\alpha, \beta, K, c_0, d, p)$ such that $C [(T-S)^{p/2} + (T-S)] \leq 1/2$ if $0 \leq T-S \leq c_1$. In that case

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |Y_t^{n,m}|^p \right] \leq 2C \left\{ \mathbf{E} [|Y_S^{n,m}|^p] + l(n, \beta, \alpha, p) + l(m, \beta, \alpha, p) \right\}.$$

Rate of convergence. Now let us assume Y_t is a strong solution to (1.4). We have, by Itô formula and (2.4), for $t \in [S, T]$,

$$\begin{aligned} &u(t, Y_t) - u(S, Y_S) \\ &= \int_S^t b(Y_r) dr + \int_S^t \int_{|y| \leq 1} [u(r, Y_{r-} + G(Y_{r-})y) - u(r, Y_{r-})] q(dr, dy). \end{aligned}$$

Hence for $t \in [S, T]$, we obtain

$$\begin{aligned} &Y_t - Y_S = u(t, Y_t) - u(S, Y_S) \\ &+ \int_S^t \int_{|y| \leq 1} \{G(Y_{r-})y - [u(r, Y_{r-} + G(Y_{r-})y) - u(r, Y_{r-})]\} q(dr, dy), \end{aligned}$$

and thus

$$\begin{aligned} &Y_t^n - Y_t \\ &= \{Y_S^n - Y_S + [u(t, Y_t^n) - u(S, Y_S^n)] - [u(t, Y_t) - u(S, Y_S)]\} \\ &+ \sum_{k=2}^4 B_t^{n,k} + B_t^{n,5} \\ &- \int_S^t \int_{|y| \leq 1} \{G(Y_{r-})y - [u(r, Y_{r-} + G(Y_{r-})y) - u(r, Y_{r-})]\} q(dr, dy). \end{aligned}$$

Estimates for $B_t^{n,k}, k = 2, 3, 4$ have been derived above. And we can estimate

$$\begin{aligned} &Y_S^n - Y_S + [u(t, Y_t^n) - u(S, Y_S^n)] - u(t, Y_t) - u(S, Y_S), \\ &B_t^{n,6} - \int_S^t \int_{|y| \leq 1} \{G(Y_{r-})y - [u(r, Y_{r-} + G(Y_{r-})y) - u(r, Y_{r-})]\} q(dr, dy) \end{aligned}$$

in exactly the same way as we estimated $B_t^{n,m;1}$ and $B_t^{n,m;5}$ (by replacing Y_t^m by Y_t in the arguments). We find that there is a constant $C =$

$C(\alpha, \beta, p, K, c_0, d)$ such that

$$\mathbf{E} \left[\sup_{S \leq t \leq T} |Y_t^n - Y_t|^p \right] \leq C[\mathbf{E}[|Y_S^n - Y_S|^p] + l(n, \beta, \alpha, p)],$$

and the claimed rate of convergence holds by Lemma 11. \square

The existence and uniqueness part is a simple repeat of the arguments in the proof of Proposition 1.

4. APPENDIX

We will be using some general estimates of stochastic integrals. We start with Lenglart's inequality (see [7]). Let Z_t be a nonnegative càdlàg process and A_t be an increasing predictable process. We say that A dominates Z if for any finite stopping time τ

$$\mathbf{E} Z_\tau \leq \mathbf{E} A_\tau.$$

The following moment estimate holds.

Lemma 9. (see Corollary II in [7]) *Let Z be dominated by A . Then for every $p \in (0, 1)$ and every stopping time τ ,*

$$\mathbf{E} \left[\left(\sup_{s \leq \tau} |Z_s| \right)^p \right] \leq \frac{2-p}{1-p} \mathbf{E} [A_\tau^p].$$

Remark 1. Let $H : [0, 1) \times \Omega \times \mathbf{R}_0^d \rightarrow \mathbf{R}^m$ be a $\mathcal{P} \times \mathcal{B}(\mathbf{R}_0^d)$ -measurable function, $H := H_r(y)$, $r \in [0, 1)$, $y \in \mathbf{R}^d$. Assume that for any $T \in [0, 1)$ a.s.,

$$\int_0^T \int |H_r(y)|^2 \rho(y) \frac{dy dr}{|y|^{d+\alpha}} < \infty,$$

where \mathcal{P} is a predictable σ -algebra on $[0, 1) \times \Omega$. Then

(i) (see [7])

$$Z_t = \left| \int_0^t \int H_r(y) q(dr, dy) \right|^2, t \in [0, 1),$$

is dominated by

$$A_t = \int_0^t \int |H_r(y)|^2 \rho(y) \frac{dy}{|y|^{d+\alpha}} dr, t \in [0, 1).$$

Hence by Lemma 9 (Corollary II in [7]), for any $p \in (0, 2)$ there is $C = C(p)$ such that for any stopping time τ ,

$$\begin{aligned} & \mathbf{E} \left[\sup_{t \leq \tau} \left| \int_0^t \int H_r(y) q(dr, dy) \right|^p \right] \\ & \leq C \mathbf{E} \left[\left(\int_0^\tau \int |H_r(y)|^2 \rho(y) \frac{dy}{|y|^{d+\alpha}} dr \right)^{p/2} \right]. \end{aligned}$$

(ii) On the other hand, for $p \in [1, 2]$, by BGD inequality,

$$\begin{aligned} \mathbf{E} \left[\sup_{t \leq \tau} \left| \int_0^t \int H_r(y) q(dr, dy) \right|^p \right] &\leq C \mathbf{E} \left[\left(\int_0^\tau \int |H_r(y)|^2 \rho(y) N(dr, dy) \right)^{p/2} \right] \\ &\leq C \mathbf{E} \left[\int_0^\tau \int |H_r(y)|^p \rho(y) N(dr, dy) \right] \\ &\leq C \mathbf{E} \int_0^\tau \int |H_r(y)|^p \frac{dy}{|y|^{d+\alpha}} dr. \end{aligned}$$

Remark 2. Let $H : [0, 1) \times \Omega \times \mathbf{R}_0^d \rightarrow \mathbf{R}^m$ be a $\mathcal{P} \times \mathcal{B}(\mathbf{R}_0^d)$ -measurable function, $H := H_r(y)$, $r \in [0, 1)$, $y \in \mathbf{R}^d$, such that for any $T \in [0, 1)$ a.s.,

$$\int_0^T \int |H_r(y)| \rho(y) \frac{dy dr}{|y|^{d+\alpha}} < \infty.$$

(i) Obviously,

$$Z_t = \left| \int_0^t \int H_r(y) q(dr, dy) \right|, t \in [0, 1),$$

is dominated by

$$A_t = 2 \int_0^t \int |H_r(y)| \rho(y) \frac{dy}{|y|^{d+\alpha}} dr, t \in [0, 1).$$

Hence by Lemma 9 (Corollary II in [7]), for any $p \in (0, 1)$ there is $C = C(p)$ such that for any stopping time τ ,

$$\mathbf{E} \left[\sup_{t \leq \tau} \left| \int_0^t \int H_r(y) q(dr, dy) \right|^p \right] \leq C \mathbf{E} \left[\left(\int_0^\tau \int |H_r(y)| \rho(y) \frac{dy}{|y|^{d+\alpha}} dr \right)^p \right].$$

(ii) For $p \in [1, 2]$, by BDG inequality, we have as in (4.1),

$$\begin{aligned} \mathbf{E} \left[\sup_{t \leq \tau} \left| \int_0^t \int H_r(y) q(dr, dy) \right|^p \right] &\leq C \mathbf{E} \left[\left(\int_0^\tau \int |H_r(y)|^2 \rho(y) N(dr, dy) \right)^{p/2} \right] \\ &\leq C \mathbf{E} \int_0^\tau \int |H_r(y)|^p \frac{dy}{|y|^{d+\alpha}} dr. \end{aligned}$$

For the sake of completeness we remind two other “general” estimates.

Lemma 10. (see e.g. Lemma 4.1 in [6]) (i) (Kunita’s inequality) Let $H : [0, 1) \times \Omega \times \mathbf{R}_0^d \rightarrow \mathbf{R}^m$ be a $\mathcal{P} \times \mathcal{B}(\mathbf{R}_0^d)$ -measurable function, $H := H_r(y)$, $r \in [0, 1)$, $y \in \mathbf{R}^d$, such that for any $T \in [0, 1)$ a.s.,

$$\int_0^T \int |H_r(y)|^2 \rho(y) \frac{dy dr}{|y|^{d+\alpha}} < \infty,$$

where \mathcal{P} is a predictable σ -algebra on $[0, 1) \times \Omega$. Then for each $p \geq 2$ there is $C = C(p)$ such that for any stopping time τ ,

$$\begin{aligned} \mathbf{E} \left[\sup_{t \leq \tau} \left| \int_0^t \int H_r(y) q(dr, dy) \right|^p \right] &\leq C \mathbf{E} \left[\int_0^\tau \int |H_r(y)|^p \rho(y) \frac{dy}{|y|^{d+\alpha}} dr \right] \\ &\quad + C \mathbf{E} \left[\left(\int_0^\tau \int |H_r(y)|^2 \rho(y) \frac{dy}{|y|^{d+\alpha}} dr \right)^{p/2} \right]. \end{aligned}$$

(ii) Let $H : [0, 1) \times \Omega \times \mathbf{R}_0^d \rightarrow \mathbf{R}^m$ be a $\mathcal{P} \times \mathcal{B}(\mathbf{R}_0^d)$ -measurable function, $H := H_r(y)$, $r \in [0, 1)$, $y \in \mathbf{R}^d$, such that for any $T \in [0, 1)$ a.s.,

$$\int_0^T \int |H_r(y)| \rho(y) \frac{dy dr}{|y|^{d+\alpha}} < \infty,$$

Then for each $p \geq 1$ there is $C = C(p)$ such that for any stopping time τ ,

$$\begin{aligned} \mathbf{E} \left[\sup_{t \leq \tau} \left| \int_0^t \int H_r(y) N(dr, dy) \right|^p \right] &\leq C \mathbf{E} \left[\int_0^\tau \int |H_r(y)|^p \rho(y) \frac{dy}{|y|^{d+\alpha}} dr \right] \\ &\quad + C \mathbf{E} \left[\left(\int_0^\tau \int |H_r(y)| \rho(y) \frac{dy}{|y|^{d+\alpha}} dr \right)^p \right]. \end{aligned}$$

Remark 3. Let $H : [0, 1) \times \Omega \times \mathbf{R}_0^d \rightarrow \mathbf{R}^m$ be a $\mathcal{P} \times \mathcal{B}(\mathbf{R}_0^d)$ -measurable function, $H := H_r(y)$, $r \in [0, 1)$, $y \in \mathbf{R}^d$, such that for any $T \in [0, 1)$ a.s.,

$$\int_0^T \int |H_r(y)| \rho(y) \frac{dy dr}{|y|^{d+\alpha}} < \infty.$$

(i) Since $N(dr, dy)$ -stochastic integral is a sum, a.s. for every $p \in (0, 1)$, $t \in [0, 1)$,

$$\left| \int_0^t \int H_r(y) N(dr, dy) \right|^p \leq \int_0^t \int |H_r(y)|^p N(dr, dy).$$

Hence for any stopping time τ ,

$$\mathbf{E} \left[\sup_{t \leq \tau} \left| \int_0^t \int H_r(y) N(dr, dy) \right|^p \right] \leq \mathbf{E} \int_0^\tau \int |H_r(y)|^p \rho(y) \frac{dy dr}{|y|^{d+\alpha}}.$$

(ii) On the other hand, $Z_t = \left| \int_0^t \int H_r(y) N(dr, dy) \right|$, $t \in [0, 1)$, is obviously dominated by

$$A_t = \int_0^t \int |H_r(y)| \rho(y) \frac{dy dr}{|y|^{d+\alpha}}, t \in [0, 1).$$

By Lemma 9, for each $p \in (0, 1)$, there is $C = C(p) > 0$ so that

$$\mathbf{E} \left[\sup_{t \leq \tau} \left| \int_0^t \int H_r(y) N(dr, dy) \right|^p \right] \leq C \mathbf{E} \left[\left(\int_0^\tau \int |H_r(y)| \rho(y) \frac{dy dr}{|y|^{d+\alpha}} \right)^p \right]$$

We use the following simple statement about derivation of a global estimate from a local one.

Lemma 11. *Let $Z_t, t \in [0, 1]$, be a nonnegative càdlàg stochastic process, $Z_0 = 0$ and $p > 0$. Assume there is $\delta \in (0, 1)$ and $N, L > 0$ such that for any $0 \leq S \leq T < 1$ with $|T - S| \leq \delta$, we have*

$$\mathbf{E} \left[\sup_{S \leq t \leq T} Z_t^p \right] \leq N[\mathbf{E}[Z_S^p] + L].$$

Then there is $C = C(\delta, L, N)$ so that

$$\mathbf{E} \left[\sup_{0 \leq t \leq 1} Z_t^p \right] \leq CL.$$

Proof. We partition $[0, 1]$ into N_0 subintervals of length $N_0^{-1} \leq \delta$. Let $S_k = k/N_0, k = 0, \dots, N_0$, and

$$A_k = \mathbf{E} \left[\sup_{S_{k-1} \leq t \leq S_k} Z_t^p \right], k = 1, \dots, N_0.$$

then,

$$\begin{aligned} A_k &\leq NA_{k-1} + NL, k = 2, \dots, N_0, \\ A_1 &\leq NL, \end{aligned}$$

and then

$$A_k \leq (N^k + \dots + N) L = C_k L, k = 1, \dots, N_0.$$

Therefore,

$$\mathbf{E} \left[\sup_{0 \leq t \leq 1} Z_t^p \right] \leq (C_1 + \dots + C_{N_0})L.$$

□

REFERENCES

- [1] Applebaum, D., Lévy Processes and Stochastic Calculus, 2nd edition, Cambridge University Press, 2009.
- [2] DaPrato, G., Flandoli, F., Priola, E., and Röckner, M., Strong uniqueness for stochastic evolution equations in Hilbert spaces with bounded and measurable drift, Annals of Probability, 41(5), 2013, pp 3306-3344.
- [3] Fedrizzi, F. and Flandoli, F., Pathwise uniqueness and continuous dependence of SDEs with non-regular drift, Stochastics, 83, 2011, pp 241-257.
- [4] Flandoli, F., Gubinelli, M. and Priola, E., Well-posedness of the transport equation by stochastic perturbation, Invent. Math., 180, 2010, pp 1-53.
- [5] Jacod, J., The Euler scheme for Levy driven SDEs: limit theorems, Annals of Probab., 32(3), 2004, pp 1830-1872.
- [6] Leahy, J.-M. and Mikulevicius, R., On classical solutions of linear stochastic integro-differential equations, Stoch PDE: Anal Comp, DOI10.1007/s40072-016-0070-5, 2016, pp 1-57.
- [7] Lenglart, E., Relation de domination entre deux processus, Annales de l'I.H.P., section B, 13(2), 1977, pp 171-179.

- [8] Mikulevicius, R., and Pragarauskas, H., On the Cauchy problem for integro-differential operators in Hölder classes and the uniqueness of the martingale problem, *Potential Analysis*, 40(4), 2014, pp 539-563.
- [9] Pamen, O.M., and Taguchi, D., Strong rate of convergence for the Euler-Maruyama approximation of SDEs with Hölder continuous drift coefficient, *arXiv:1508.07513*, 2016, pp 1-19.
- [10] Priola, E., Pathwise uniqueness for singular SDEs driven by stable processes, *Osaka J. Math.* 49(2), 2012, pp 421-447.

UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES